Aufgaben zur Topologie I

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Jean-Pierre Serre, *1926, french mathematician

Exercise 4.1 (Quasi-isomorphisms)
Let $\iota\colon A \to B$ be the inclusion of a subcomplex A of a chain complex B. Assume the following:
For each n and any $b \in B_n$ there exists some $a \in A_n$ and some $b' \in B_{n+1}$ such that $\iota(a) = b + \partial(b')$.
Show that ι induces an ismorphism in homology $\iota_* = H_n(\iota)\colon H_n(A) \to H_n(B)$ for all n.
Remark: A chain map which induces isomorphisms in all homology groups is called a *quasi-isomorphism*.

Exercise 4.2 (Free and projective modules)
Let $\mathbb{K}$ be a commutative ring with unit. A $\mathbb{K}$-module $\mathbb{M}$ is called *free* if it has a basis $\mathfrak{B} \subset \mathbb{K}$, this means, each $x \in \mathbb{M}$ can be expressed as $x = \kappa_1 b_1 + \ldots + \kappa_n b_n$ for finitely many uniquely determined basis elements $b_1, \ldots, b_n \in \mathfrak{B}$ and coefficients $\kappa_1, \ldots, \kappa_n \in \mathbb{K}$.

1. A free module is projective.
2. Any module is a quotient of a free (and thus of a projective) module.
3. Any submodule $\mathbb{L} \subset \mathbb{M}$ with a projective quotient $\mathbb{M}/\mathbb{L}$ is a direct summand.

Let $\phi\colon \mathbb{K} \to \mathbb{K}'$ be a ring-homomorphism; it makes $\mathbb{K}'$ into a $\mathbb{K}$-module.

4. If $\mathbb{M}$ is a $\mathbb{K}$-module, then $\mathbb{M}' := \mathbb{M} \otimes_{\mathbb{K}} \mathbb{K}'$ is a $\mathbb{K}'$-module.
5. If $\mathbb{M}$ is free over $\mathbb{K}$, then $\mathbb{M}' := \mathbb{M} \otimes_{\mathbb{K}} \mathbb{K}'$ is free over $\mathbb{K}'$.

Let $\mathbb{K} = \mathbb{Z}$ and let $\phi_n\colon \mathbb{Z} \to \mathbb{Z}/n$ be the obvious epimorphism. Clearly, any $\mathbb{Z}/n$-module is also a $\mathbb{Z}$-module, but a $\mathbb{Z}$-module is a a $\mathbb{Z}/n$-module if and only if $nx = 0$ holds for any module element x.

6. $\mathbb{M} = \mathbb{Z}/n$ is free over $\mathbb{Z}/n$, but not over $\mathbb{Z}$ and not over any $\mathbb{Z}/nm$ for $m > 1$.

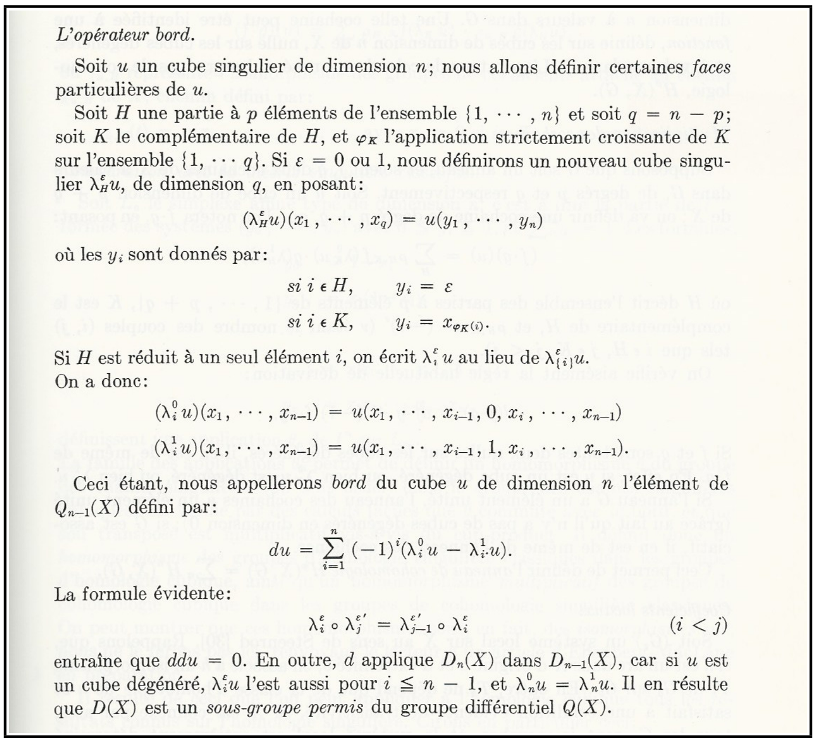
Exercise 4.3 (Acyclic vs. contractible chain complexes)
a) Let $\mathbb{K}$ be a commutative ring, and let P be a non-negatively graded chain complex of projective $\mathbb{K}$-modules P_n. Assume that P is acyclic, i.e., all its homology groups are trivial. Show that P is contractible.

b) Give an example of a non-negatively graded chain complex of abelian groups, which is acyclic, but not contractible.

c) Show that the unbounded chain complex of projective $\mathbb{Z}/4$-modules

$$\ldots \xrightarrow{2\cdot} \mathbb{Z}/4 \xrightarrow{2\cdot} \mathbb{Z}/4 \xrightarrow{2\cdot} \mathbb{Z}/4 \xrightarrow{2\cdot} \ldots$$

is acyclic, but not contractible.



L'opérateur bord.

Soit u un cube singulier de dimension n; nous allons définir certaines *faces* particulières de u.

Soit H une partie à p éléments de l'ensemble $\{1, \cdots, n\}$ et soit $q = n - p$; soit K le complémentaire de H, et φ_K l'application strictement croissante de K sur l'ensemble $\{1, \cdots q\}$. Si $\varepsilon = 0$ ou 1, nous définirons un nouveau cube singulier $\lambda_H^\varepsilon u$, de dimension q, en posant:

$$(\lambda_H^\varepsilon u)(x_1, \cdots, x_q) = u(y_1, \cdots, y_n)$$

où les y_i sont donnés par:

$$\begin{aligned} &si\ i \in H, \qquad y_i = \varepsilon \\ &si\ i \in K, \qquad y_i = x_{\varphi_K(i)}. \end{aligned}$$

Si H est réduit à un seul élément i, on écrit $\lambda_i^\varepsilon u$ au lieu de $\lambda_{\{i\}}^\varepsilon u$.
On a donc:

$$(\lambda_i^0 u)(x_1, \cdots, x_{n-1}) = u(x_1, \cdots, x_{i-1}, 0, x_i, \cdots, x_{n-1})$$

$$(\lambda_i^1 u)(x_1, \cdots, x_{n-1}) = u(x_1, \cdots, x_{i-1}, 1, x_i, \cdots, x_{n-1}).$$

Ceci étant, nous appellerons *bord* du cube u de dimension n l'élément de $Q_{n-1}(X)$ défini par:

$$du = \sum_{i=1}^{n} (-1)^i (\lambda_i^0 u - \lambda_i^1 u).$$

La formule évidente:

$$\lambda_i^\varepsilon \circ \lambda_j^{\varepsilon'} = \lambda_{j-1}^{\varepsilon'} \circ \lambda_i^\varepsilon \qquad (i < j)$$

entraîne que $ddu = 0$. En outre, d applique $D_n(X)$ dans $D_{n-1}(X)$, car si u est un cube dégénéré, $\lambda_i^\varepsilon u$ l'est aussi pour $i \leqq n - 1$, et $\lambda_n^0 u = \lambda_n^1 u$. Il en résulte que $D(X)$ est un *sous-groupe permis* du groupe différentiel $Q(X)$.

J.-P. Serre: *Homologie singuliere des espaces fibres. Applications.* Ann. Math. 54 (1951), 425-505, here page 440. This is one of the most important articles in algebraic topology, by one of its greatest masters.

Exercise 4.4 (Wrong cubic homology)
We consider continuous maps $c\colon \mathbb{I}^n \to X$ from the n-cube $\mathbb{I}^n = \mathbb{I} \times \cdots \times \mathbb{I}$, and $\mathbb{I} = [0,1]$ being the interval, to a space X and call them *singular n-cubes* in X. They form the basis for the free $\mathbb{Z}$-module $K_n(X)$. For $n < 0$ we set $K_n(X) = 0$; and $\mathbb{I}^0$ is just a point. Define face maps $d_i^0, d_i^1\colon \mathbb{I}^{n-1} \longrightarrow \mathbb{I}^n$ by

$$d_i^0(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

and

$$d_i^1(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 1, t_i, \ldots, t_{n-1}).$$

They induce face operators $\partial_i^0, \partial_i^1\colon K_n(X) \longrightarrow K_{n-1}(X)$ by setting $\partial_i^0(c) := c \circ d_i^0$ resp. $\partial_i^1(c) := c \circ d_i^1$ for a basis element $c \in K_n(X)$.

(i) Prove the following cubical identities:

1. $\partial^b_{j-1} \circ \partial^a_i = \partial^a_i \circ \partial^b_j \quad$ for $0 \leq i < j \leq n$ and $a, b \in \{0, 1\}$,

2. .... ,

3. ....

where the dots mean: if you like, define degeneracy maps $s^0_i, s^1_i \colon \mathbb{I}^n \to \mathbb{I}^{n-1}$ and prove relations between any two of them and with the face maps.
Define a boundary operator $\partial \colon K_n(X) \longrightarrow K_{n-1}(X)$ by

$$\partial = \sum_{i=0}^{n} (-1)^i \, (\partial^1_i - \partial^0_i) \, .$$

(i) Prove $\partial \circ \partial = 0$.
So we have a chain complex $K_\bullet(X)$. Its homology we call *wrong cubical homology* $\mathrm{WH}^\square_n(X) := H_n(K_\bullet(X))$.
What is wrong about it ? — Well, see yourself:

(ii) Prove for X a point: $\mathrm{WH}^\square_n(X) = \mathbb{Z}$ for each $n \geq 0$.
(Hint: Do the computation similarly to the simplicial case: What are all singular cubes ? What is their boundary ?) The result is not what we expected, since the dimension axiom is not satisfied; we will see later how to correct this.

(iii) Show that any continuous map $f \colon X \to Y$ induces a chain map $K_n(f) \colon K_n(X) \to K_n(Y)$ and thus a homomorphism $\mathrm{WH}^\square_n(f) \colon \mathrm{WH}^\square_n(X) \to \mathrm{WH}^\square_n(Y)$ between homology groups.

(iv) Show further, that $\mathrm{WH}^\square_n$ is a functor from the category of topological spaces to the category of $\mathbb{K}$-modules.

(v) Prove: If $f \simeq g \colon X \to Y$ are homotopic, then $K_n(f) \simeq K_n(g)$ are chain homotopic.
Conclude that $\mathrm{WH}^\square_n(f) = \mathrm{WH}^\square_n(g)$.

Exercise 4.5 (The chain complex of chain functions)
Let C and D be two chain complexes over the ring $\mathbb{K}$, with boundary operators ∂^C resp. ∂^D. We define a *chain function f of degree k* to be a collection of homomorphisms $f_n \colon C_n \to D_{n+k}$. Note that we do not assume any compatibility with the boundary operators; also note that k can be negative. Obviously, the chain functions of degree k form a $\mathbb{K}$-module $F_k := \mathrm{ChFunc}_k(C, D)$.
Furthermore, by declaring $d(f)$ to be the chain function with

$$d(f)_n := \partial^D \circ f_n - (-1)^k f_{n-1} \circ \partial^C \qquad \text{for all } n,$$

we obtain a homomorphism

$$d \colon F_k \longrightarrow F_{k-1}$$

a) Show that $d \circ d = 0$, so that F with this boundary operator is a chain complex.
b) Show that $Z_0(F)$, the cycles of degree 0, are the *chain maps* (i.e., they satisfy $\partial^D \circ f_n = f_{n-1} \circ \partial^C$).
c) Show that $B_0(F)$, the boundaries of degree 0, are all chain maps homotopic to zero.
d) Show that $H_0(F)$ are the chain homotopy classes of chain maps $C \to D$.
e*) What are $Z_n(F)$, $B_n(F)$ and $H_n(F)$ for $n > 0$?

Exercise 4.6* (Cycles and geometric intuition)
Let X be a space and $c = \sum_\alpha \mu_\alpha c_\alpha$ an n-cycle. The index α is in some finite index set $\mathcal{A}$. The coefficients μ_α are integers and we can assume $\mu_\alpha \neq 0$. Written out in basic chains with sign ± 1, we have altogether $r := \sum_{\alpha \in \mathcal{A}} |\mu_\alpha|$ terms.
We take, for each $\alpha \in \mathcal{A}$, exactly $|\mu_\alpha|$ copies of an n-simplex and denote it by $\Delta(\alpha, k)$, where $k = 1, \ldots, |\mu_\alpha|$. Altogether there are r such simplices and we put them together to form a space $\tilde{P}(c) = \bigsqcup \Delta(\alpha, k)$. We can regard c

as a continuous map $\tilde{f}_c\colon \tilde{P}(c) \to X$, defined by c_α on each $\Delta(\alpha,k)$. Each simplex has $n+1$ faces, which we denote by $\Delta_i(\alpha,k)$, where $i = 0,1,\ldots,n$.
From the cycle condition

$$0 = \partial(a) = \sum_{\mathcal{A}} \sum_{i=0}^{n} (-1)^i\, \partial_i(c_\alpha) = \sum_{\mathcal{A}} \sum_{i=0}^{n} (-1)^i\, (c_\alpha \circ d_i) \tag{1}$$

we conclude that for all these $r(n+1)$ faces this sum must, via their signs, cancel in $S_{n-1}(X)$. This means for any basic (n-1)-chain $b \in \mathcal{B}_{n-1}(X)$ the following: Set $\mathfrak{J}(b) := \{(\alpha,i) \in \mathcal{A} \times [n] \,|\, c_\alpha \circ d_i = b\}$, where $[n]$ denotes the set $\{0,1,\ldots,n\}$. For almost all b the set $\mathfrak{J}(b)$ must be empty. For all others

$$\sum_{(\alpha,i)\in\mathfrak{J}(b)} (-1)^i \mu_\alpha = 0 \tag{2}$$

must hold. In other words, for each b, we have

$$\sum_{(\alpha,i)\in\mathfrak{J}(b),\,(-1)^i\mu_\alpha>0} |\mu_\alpha| \quad = \quad \sum_{(\alpha,i)\in\mathfrak{J}(b),\,(-1)^i\mu_\alpha<0} |\mu_\alpha|\ , \tag{3}$$

when we split the sum in positive and negative coefficients $(-1)^i\mu_\alpha$. So the two sets $K^+(b)$ resp. $K^-(b)$ of triples (α,k,i) with $b = c_\alpha \circ d_i$, $k = 1,\ldots,|\mu_\alpha|$ and $(-1)^i \operatorname{sign}(\mu_\alpha) = +1$ resp. with $b = c_\alpha \circ d_i$, $k = 1,\ldots,|\mu_\alpha|$ and $(-1)^i \operatorname{sign}(\mu_\alpha) = -1$ have the same size and we can choose a bijection $\pi_b\colon K^+(b) \to K^-(b)$ with the property

$$(-1)^i \operatorname{sign}(\mu_\alpha)\, \partial_i(c_\alpha) = (-1)^j \operatorname{sign}(\mu_\beta)\, \partial_j(c_\beta), \quad \text{if} \quad \pi_b(\alpha,k,i) = (\beta,l,j) \tag{4}$$

Note that k and l do not occur in the equation. And note that there are many choices for such a bijection or pairing. We take the disjoint union of all $K^+(b)$ resp. of all $K^-(b)$ and call them K^+ resp. K^-. The obvious bijection we call $\pi\colon K^+ \to K^-$.
Now recall $\tilde{P}(c) = \bigsqcup \Delta(\alpha,k)$. On each simplex we took the continous map $c_\alpha\colon \Delta(\alpha,k) \to X$. It follows from the equation above, that c_α and c_β agree on their faces $\Delta_i(\alpha,k)$ resp. $\Delta_j(\alpha,l)$, if $\pi_b(\alpha,k,i) = (\beta,l,j)$.
Thus in $\tilde{P}(c)$ we can identify the two faces $\Delta_i(\alpha,k)$ and $\Delta_j(\beta,l)$ by declaring

$$\Delta_i(\alpha,k) \ni (t_0,\ldots,t_{i-1},0,t_i,\ldots,t_{n-1}) \equiv (t_0,\ldots,t_{j-1},0,t_j,\ldots,t_{n-1}) \in \Delta_j(\beta,k). \tag{5}$$

Call this space $P(c,\pi)$ the *tautological complex* of the cycle c with pairing π. We have a well-defined map

$$f_c\colon P(c,\pi) \longrightarrow X,$$

which is c_α on each $\Delta(\alpha,k)$.
Now we want to investigate this space and this map.

(1) Show by an example with $n = 1$ that $P(c,\pi)$ depends on the choice of the pairing π.

(2) Show that c gives rise to a canonical n-cycle w_c in $P(c,\pi)$. Its homology class $[w_c] \in H_n(P(c,\pi))$ we call the *tautological class* of $P(c,\pi)$.

(3) Show that $f_{c*}([w_c]) = [c]$ in $H_n(X)$.

Remarks: (1) $P(c,\pi)$ is a space for which one can define simplicial homology, as we did in Exercise 2.4. Obviously, we can regards w_c as a simplicial cycle; and in $H_n^\triangle(P(c,\pi))$ the class $[w_c]$ is non-zero, because there are no simplicial (n+1)-chains to kill it. Later we will see that the natural transformation $H_n^\triangle \to H_n$ from simplicial to singular homology is an isomorphism; thus $[w_c]$ is non-zero in $H_n(P(c,\pi)$. But of course, this does not mean, that $f_c[w–c] = [c]$, is non-zero.
(2) The space $P(c,\pi)$ is the union of n-simplices and each (n-1)-simplex is in exactly two n-simples (related by the pairing π). One might think that $P(c,\pi)$ is a manifold; but this is not the case. Nevertheless, it has many features of a manifold and is called a pseudo-manifold.
(3) What about $\mathbb{Z}/2$ as coefficients ? What about $\mathbb{Z}/3$?