Aufgaben zur Topologie I

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Blatt 3 due by: 30. 10. 2019

In the figures, the strand going under a crossing is graphically represented by a line broken near the crossing; the strand going over a crossing is represented by a continued line. An example of a braid diagram is given in Figure 1.2. Here the top horizontal line represents $\mathbf{R} \times \{0\}$, and the bottom horizontal line represents $\mathbf{R} \times \{1\}$. In the sequel we shall sometimes draw and sometimes omit these lines in the figures.

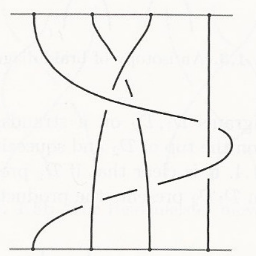


Fig. 1.2. A braid diagram on four strands

Exercise 3.1 (Homotopic versus homologous)
Let w and v are two closed curves in a space X.

(1) Assume that w and v are freely homotopic in X. This means: $w(0) = w(1) = x_0$ and $v(0) = v(1) = y_0$, but x_0 and y_0 may be distinct and thus x_0 must be moved towards y_0 during the homotopy. Show that w and v are homologous, when regarded as 1-cycles in X.

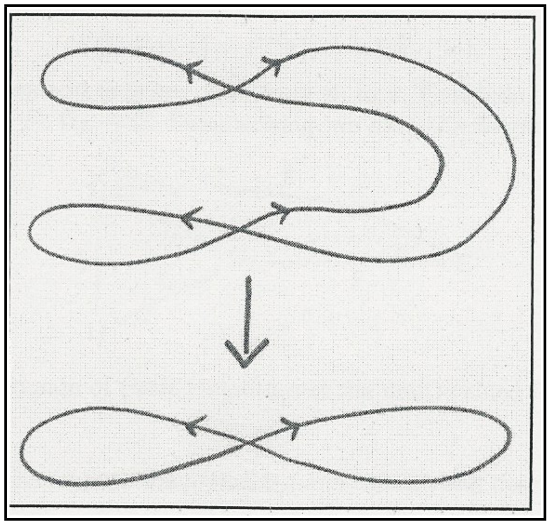
(2) Assume w and v are in the same path-component of X. Show that there exists a closed curve c in X, such that c is homologous to $w + v$, when regarded as 1-cycles.

(3) Find an example of two closed curves w and v, which are homologous (when regarded as 1-cycles), but are not homotopic. (Hint: Consider a surface.)

Exercise 3.2 (Coverings and first homology group)
Let $\xi\colon X \to Y$ be a connected covering of a connected space. Recall that the induced homomorphism of fundamental groups $\pi_1(\xi)\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is injective. Is the same true for $H_1(\xi)\colon H_1(X) \to H_1(Y)$? Consider the covering of $Y = \mathbb{S}^1 \vee \mathbb{S}^1$ shown in the following figure.

1) Compute $H_1(X$ and $H_1(Y)$ using the Hurewicz isomorphism.

2) Compute $H_1(\xi)$ using the Hurewicz isomorphism and conclude that the answer to the question above is not always Yes.



A 2-fold covering of the space $Y = \mathbb{S}^1 \vee \mathbb{S}^1$, used in Exerc. 3.2.

Exercise 3.3 (Induced homomorphisms)

(i) Let X, Y be topological spaces and $\iota\colon X \to Y$, $\rho\colon Y \to X$ continuous maps, which satisfy $\rho \circ \iota = \mathrm{id}_X$, i.e., X is a *retract* of Y. Show that, for every $n \in \mathbb{N}$, the homology $H_n(X)$ of X is isomorphic to a direct summand of the homology $H_n(Y)$ of Y.

(ii) Let $f\colon X \to Y$ be a constant function. Show that $H_n(f)\colon H_n(X) \to H_n(Y)$ is the trivial homomorphism for all $n > 0$. How can one describe $H_0(f)$?

(iii) Describe for a general $f\colon X \to Y$ the induced map $H_0(f)\colon H_0(X) \to H_0(Y)$.

(iv) Compute, with the help of the Hurewicz isomorphism, the homomorphism

$$\mathbb{Z} \cong H_1(\mathbb{S}^1) \xrightarrow{H_n(f)} H_1(\mathbb{S}^1 \vee \mathbb{S}^1) \cong (\mathbb{Z} \star \mathbb{Z})^{ab} \cong \mathbb{Z} \times \mathbb{Z}$$

induced by the pinch map $\mathbb{S}^1 \to \mathbb{S}^1 \vee \mathbb{S}^1$, which identifies 1 and -1.

Exercise 3.4 (Some small chain complexes)
a) Consider the following chain complex $A_\bullet$

$$\ldots \longleftarrow 0 \longleftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z}^3 \xleftarrow{\partial} \mathbb{Z}^2 \longleftarrow 0 \longleftarrow \ldots$$

All non-trivial terms are shown, the term $\mathbb{Z}$ sits in degree 0, and boundary operator is written as matrix $\partial = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}$. Compute all homology groups.

b) Consider the following chain complex $B_\bullet$

$$\ldots \longleftarrow 0 \longleftarrow \mathbb{Z}/2 \xleftarrow{\partial_1} \mathbb{Z}/2 \oplus \mathbb{Z}/6 \xleftarrow{\partial_2} \mathbb{Z} \oplus \mathbb{Z}/4 \xleftarrow{\partial_3} \mathbb{Z} \longleftarrow 0 \longleftarrow \ldots$$

All non-trivial terms are shown, the term $\mathbb{Z}/2$ sits in degree 0, and the boundary operators are in matrix form $\partial_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $\partial_2 = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$, and $\partial_3 = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$. Compute all homology groups.

Exercise 3.5 (Chain homotopy is an equivalence relation.)
Let A, B, C be chain complexes over some ring $\mathbb{K}$, and let $f, f', f'' \colon A \to B$ and $g, g' \colon B \to C$ be chain maps. Suppose $\Phi \colon f \simeq f'$, $\Psi \colon f' \simeq f''$ and $\Theta \colon g \simeq g'$ are chain homotopies. Show that there are chain homotopies
(a) $f \simeq f$,
(b) $f' \simeq f$,
(c) $f \simeq f''$.
Furthermore,
(d) $g \circ f \simeq g' \circ f$ and
(e) $g \circ f \simeq g \circ f'$.
Thus chain homotopy is an equivalence relation and is preserved by pre-composition and post-composition.

Exercise 3.6* (Braid groups)
a) Let us first consider the symmetric group $\mathfrak{S}_n$. It is generated by the transpositions $\tau_i = (i, i+1)$ for $i = 1, 2, \ldots, n-1$ with relations

(I) $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$

(II) $\tau_i \tau_j = \tau_j \tau_i$, if $|i - j| \geq 2$

(III) $\tau_i^2 = 1$

Show that $\mathfrak{S}_n^{ab}$ is isomorphic to $\mathbb{Z}/2$. (Hint: Use the sign homomorphism $\mathrm{sign} \colon \mathfrak{S}_n \to \{\pm 1\} \cong \mathbb{Z}/2$, which has a section (right-inverse)).
b) Now we consider the braid group B_n. It is generated by *string-twists* β_i for $i = 1, 2, \ldots, n-1$ with relations as above, but without (III), namely

(I) $\beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1}$

(II) $\beta_i \beta_j = \beta_j \beta_i$, if $|i - j| \geq 2$

Show that B_n^{ab} is isomorphic to $\mathbb{Z}$. (Hint: Use the homomorphism $\widetilde{\mathrm{sign}} \colon \mathrm{B}_n \to \mathbb{Z}$, defined by $\beta_i \mapsto 1$. Is it well-defined ? Does it have a section ?)

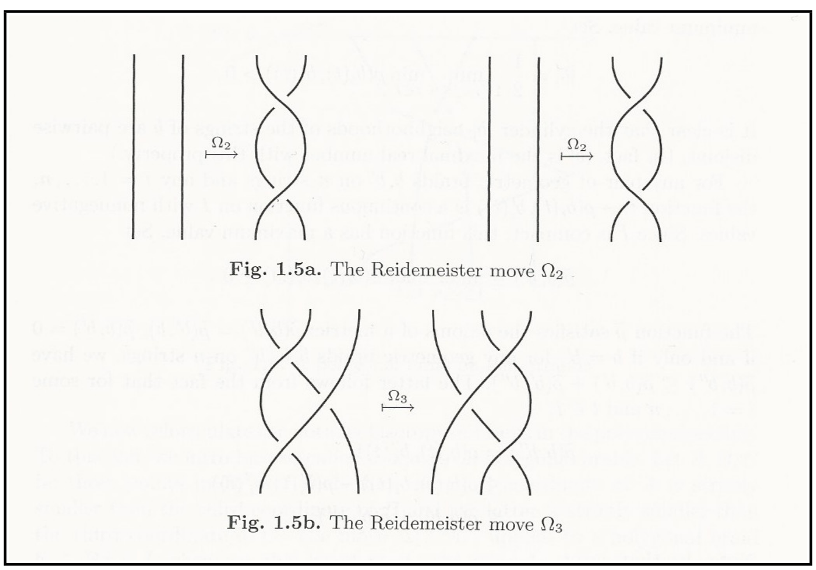


Fig. 1.5a. The Reidemeister move Ω_2

Fig. 1.5b. The Reidemeister move Ω_3

The relations in the braid group are called Reidemeister moves. This figure and the figure at the beginning are from the book Chr. Kassel, V.Turaev: *Braid Groups*, p. 7 + 9.

c)* Connect the two cases via a diagrame

$$\begin{tikzcd}
1 \arrow[r] & {[\mathrm{B}_n, \mathrm{B}_n]} \arrow[r] \arrow[d, "\Phi|"'] & \mathrm{B}_n \arrow[r, "\widetilde{\text{sign}}"] \arrow[d, "\Phi"'] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \mathfrak{A}_n \arrow[r] & \mathfrak{S}_n \arrow[r, "\text{sign}"] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

by defining a homomorphism Φ; the unnamed homomorphism on the right is the obvious one. Here, $[B_n, B_n]$ denotes the commutator subgroup of B_n. The kernel of sign is of course the well-known alternating group $\mathfrak{A}_n$. The kernel of Φ is denoted by PB_n and called the pure braid group.

Correction: In the previous version of this exercise we had mistakenly defined PB_n as the kernel of $\widetilde{\text{sign}}$ instead of Φ. This is now corrected above.

Remark: *The braid group* B_n *is usually defined as the group of isotopy classes of disjoint strings in* $\mathbb{R}^2 \times [0,1]$ *runnning (upwards) from the points* $(i, 0, 0)$ *to the points* $(j, 0, 1)$*, for* $i, j = 1, 2, \ldots, n$*. The group multiplication is given by concatenation along the third variable (like in the fundamental group). See the figure. The group is (almost obviously) the fundamental group of the unordered configuration space of* n *distinct points in* $\mathbb{R}^2$*. And by the way: The symmetric group* $\mathfrak{S}_n$ *is also a fundamental group of some unordered configuration space on* n *points, but now in* $\mathbb{R}^\infty$*. Can you see the homomorphism* Φ *?*

8.10. Theorem (Poincaré). *There is a compact 3-manifold having the homology groups of $\mathbf{S}^3$ but which is not simply connected.*

Proof. Consider the group I of rotational symmetries of a regular icosahedron, the "icosahedral group." We have $I \subset \mathbf{SO}(3)$ and it is well known that I is isomorphic to the alternating group A_5 on five letters. (This can be

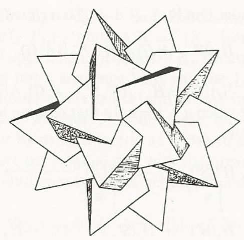


Figure VI-6. Shows that $I = A_5$.

seen geometrically by considering the five tetrahedra inscribed in a dodecahedron (which is dual to the icosahedron) and the permutations of them induced by the action of I. See Figure VI-6.) Also well known is the fact that this group is simple. Consider the homomorphism $\mathbf{S}^3 \to \mathbf{SO}(3)$, where $\mathbf{S}^3$ is the group of unit quaternions. The inverse image of I in $\mathbf{S}^3$ is a group I', of which I is the quotient by the subgroup $\{\pm 1\} \subset I'$. The dodecahedron has an inscribed cube, so that I contains the rotation group of a cube. Assuming the cube to be aligned with the coordinate axes, this implies that the quaternions i, j, k are in I'. Thus $iji^{-1}j^{-1} = ijij = k^2 = -1$ is in the commutator subgroup $[I', I']$. The image of $[I', I']$ in I is $[I, I] = I$, and it follows that $[I', I'] = I'$. The space in question is $\Sigma^3 = \mathbf{S}^3/I'$. From covering space theory, we have $\pi_1(\Sigma^3) \approx I'$ and so $H_1(\Sigma^3) \approx \pi_1(\Sigma^3)/[\pi_1, \pi_1] = 0$. By the Universal Coefficient Theorem, $H^1(\Sigma^3) \approx \mathrm{Hom}(H_1(\Sigma^3), \mathbf{Z}) = 0$. By Poincaré duality, $H_2(\Sigma^3) \approx H^1(\Sigma^3) = 0$. □

This example occupies an interesting niche in the history of topology. Poincaré originally conjectured that a manifold which is a homology sphere is homeomorphic to a sphere. When the above counterexample, called the "Poincaré dodecahedral space," and others came to light, the conjecture was modified to include the hypothesis of simple connectivity. Today, for smooth manifolds, that conjecture is known to be true with the single exception of dimension three, where it remains an open and very important conjecture called, of course, the "Poincaré Conjecture."

The 'Poincaré sphere', as explained in G.Bredon, *Topology and Geometry*, p. 353+354.