Aufgaben zur Topologie I

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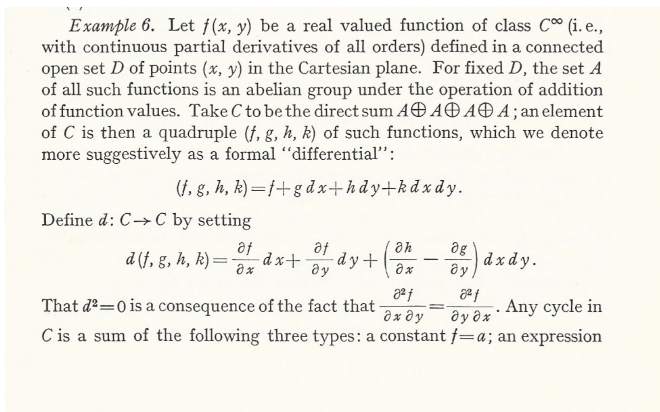
Blatt 2 due by: 23.10.2019



Example 6. Let $f(x, y)$ be a real valued function of class C^∞ (i.e., with continuous partial derivatives of all orders) defined in a connected open set D of points (x, y) in the Cartesian plane. For fixed D, the set A of all such functions is an abelian group under the operation of addition of function values. Take C to be the direct sum $A \oplus A \oplus A \oplus A$; an element of C is then a quadruple (f, g, h, k) of such functions, which we denote more suggestively as a formal "differential":

$$(f, g, h, k) = f + g\,dx + h\,dy + k\,dx\,dy.$$

Define $d\colon C \to C$ by setting

$$d(f, g, h, k) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\right) dx\,dy.$$

That $d^2 = 0$ is a consequence of the fact that $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$. Any cycle in C is a sum of the following three types: a constant $f = a$; an expression

From S.MacLane: *Homology*, p. 37.

Exercise 2.1($\mathbb{K}$-modules)
Let $\mathbb{K}$ be a commutative ring with unit 1. The rings we will consider are mainly from the following list of examples: $\mathbb{Z}$, the ring of integers; $\mathbb{Z}/n = \mathbb{Z}/n\mathbb{Z}$, the ring of integers mod n; $\mathbb{Q} \leq \mathbb{R} \leq \mathbb{C}$, the fields of rational, real resp. complex numbers. Note that, for a prime p, the ring $\mathbb{Z}/p = \mathbb{F}_p$ is also a field.
A $\mathbb{K}$-module (or a module over $\mathbb{K}$) is an abelian group A with an associative, bilinear and unital scalar action $\mathbb{K} \times A \to A$ (in analogy with the axioms of the scalar multiplication of a field on a vector space). So if $\mathbb{K}$ is a field, then A is a vector space.
Show for the other examples:
(1) Any abelian Group A is a $\mathbb{Z}$-module.
(2) An abelian group A is a $\mathbb{Z}/n$-module iff all elements $a \in A$ satisfy the equation $n\,a = a + \ldots + a$ (n summands) $= 0$.

Let $F_{\mathbb{K}}(B)$ be the free $\mathbb{K}$-module generated by the set B. Any $\mathbb{K}$-module of this form, i.e., with a basis, is called free. For free modules, show the following:

(3.1) $F_{\mathbb{K}}(\emptyset) = 0$.

(3.2) $F_{\mathbb{K}}(B_1 \sqcup B_2) \cong F_{\mathbb{K}}(B_1) \oplus F_{\mathbb{K}}(B_2)$. Thus $F_{\mathbb{K}}(B) \cong \mathbb{K}^r$ for a finite B with r elements.

(3.3) $F_{\mathbb{K}}(B)/F_{\mathbb{K}}(B') \cong F_{\mathbb{K}}(B - B')$

(3.4) $F_{\mathbb{K}}(B_1 \times B_2) \cong F_{\mathbb{K}}(B_1) \otimes_{\mathbb{K}} F_{\mathbb{K}}(B_2)$, in case you are familiar with the tensor product.

(3.5) (Universal property) For any function $\phi\colon B \to \mathbb{A}$ from a set B to a $\mathbb{K}$-module $\mathbb{A}$ there is exactly one homomorphism $f\colon F_{\mathbb{K}}(B) \to \mathbb{A}$ of $\mathbb{K}$-modules, such that $f(b) = \phi(b)$ for all $b \in B$. In other words, there is a bijection between the set of functions $\mathrm{Func}(B, \mathbb{A})$ and the of homomorphisms $\mathrm{Hom}_{\mathbb{K}}(F_{\mathbb{K}}(B), \mathbb{A})$.

Exercise 2.2 (Images and Kernels)
Let $f\colon A \to B$ be a homomorphism of $\mathbb{K}$-modules.
(a) Show that f factors into a surjection π followed by an injection ι over the image $J := \operatorname{im}(f)$:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\pi"'] & & B \\ & J \arrow[ur, "\iota"'] & \end{tikzcd}$$

(b) Show that there is an isomorphism $\ker(f \oplus g) \cong \ker(f) \oplus \ker(g)$ for the direct sum $f \oplus g\colon A \oplus B \to A' \oplus B'$ of two homomorphisms $f\colon A \to B$ and $g\colon A' \to B'$.

(c) Show that there is an isomorphism $\operatorname{im}(f \oplus g) \cong \operatorname{im}(f) \oplus \operatorname{im}(g)$ for the direct sum $f \oplus g\colon A \oplus B \to A' \oplus B'$ of two homomorphisms $f\colon A \to B$ and $g\colon A' \to B'$.

Exercise 2.3 (Simplicial identities)
For each natural number $n \geq 0$ let $[n]$ denote the set $\{0, 1, \ldots, n\}$ with the natural order $0 < 1 < \ldots < n$. For any two m, n we consider all weakly monotone functions $f\colon [m] \to [n]$. In particular, we have for $i = 0, 1, \ldots, n$ the *face maps* $d_i\colon [n-1] \to [n]$ defined by the properties (1) injective and (2) omitting i in the image. And there are for $j = 0, 1, \ldots n-1$ likewise *degeneracy maps* $s_j\colon [n] \to [n-1]$ defined by the properties (3) surjective and (4) repeating j in the image.

Comment: *The geometric interpretation explains the names: if we regard $k \in [n]$ as the vertex e_k of an n-simplex Δ^n, then a monotone function $f\colon [m] \to [n]$ as above determines an affine map $|f|\colon \Delta^m \to \Delta^n$ by affine extension. Then $|d_i|$ is the inclusion of the i-th face and $|s_j|$ is the projection onto the j-th face.*

(1) Show the following identities (of which we have seen (a) in the lecture course):

(a) $d_j \circ d_i = d_i \circ d_{j-1}$ for $i < j$

(b) $s_j \circ s_i = s_i \circ s_{j+1}$ for $i \leq j$

(c) $s_j \circ d_i = \begin{cases} d_i \circ s_{j-1} & \text{for } \ i < j, \\ \mathrm{id}_{[n]} & \text{for } i = j \text{ or } i = j+1, \\ d_{i-1} \circ s_j & \text{for } i > j+1. \end{cases}$

(2) Show that for any monotone function $f\colon [m] \to [n]$ there is a unique decomposition into face maps and degeneracy maps

$$f = d_{i_1} \circ d_{i_2} \ \circ \ldots \circ d_{i_r} \circ s_{j_1} \circ s_{j_2} \circ \ldots \circ s_{j_s}$$

for indices $0 \leq i_1 < i_2 < i_r \leq n$ and $0 \leq j_1 < j_1 < \ldots < j_s < m$ and $n = m - s + r$.

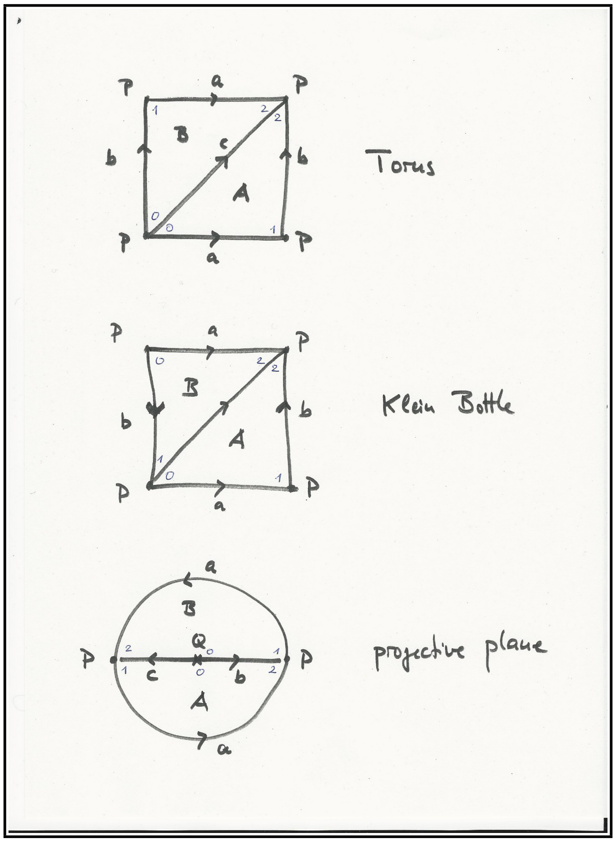
(Hint: First reduce to the special cases where f is either injective or surjective.)

Exercise 2.4 (Simplicial homology)
Let us consider a space X which is made up of vertices, edges, triangles, tetrahedra, and so on. We will not define the concept of a triangulated space (or polehedron) here; our examples are easy and the drawings are clear. They show the torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$, the Klein Bottle $\mathbb{KB}$ and the real projective plane $\mathbb{R}P^2$.

They show vertices, edges (to be identified) and triangles with their names; the vertices have with respect to each triangle they lie in a number 0, 1 or 2; this means: the edge opposite the vertex with the number i is the i-th face. For example, in the figure for the Klein Bottle the 0-th face of A is b, the 1-st face is c and the 2-nd face is a. But with respect to B the edge c is the 0-th face. Likewise, the 0-th face of an edge is its endpoint, i.e., the vertex with the larger number (note that the numbering is chosen so that each edge has a well-defined direction, the same one with respect to both of the triangles it lies in), and the 1-st face is its starting point.

To compute the (singular) homology groups of a space we need to consider the horribly large chain groups, their cycles and boundaries given by all continuous maps $\Delta^n \to X$. But is this really necessary ? — The answer is no, we can do these computations with a much smaller chain complex. Namely, it is enough to consider only the chain complex given in degree 0 by the free $\mathbb{K}$-module on the set of vertices, in degree 1 by the free $\mathbb{K}$-module on the set of edges, and in degree 2 by the free $\mathbb{K}$-module on the set of triangles. The boundary operator is defined as in the singular world, using the face maps described above. We call this chain complex the *simplicial chain complex* $S_n^{\triangle}(X)$ of X and its homology groups we denote by $H_n^{\triangle}(X;\mathbb{K})$. We will later establish an isomorphism $H_n^{\triangle}(X) \cong H_n(X)$, if X can be triangulated. But for our instant satisfaction we can do some computations already here and now.

(a) Compute the simplicial homology of the torus $\mathbb{T}$, first with coefficients $\mathbb{K} = \mathbb{Z}$ and then with $\mathbb{K} = \mathbb{Z}/2$.
(b) Compute the simplicial homology of the Klein Bottle $\mathbb{KB}$, first with $\mathbb{K} = \mathbb{Z}$ and then with $\mathbb{K} = \mathbb{Z}/2$.
(c) Compute the simplicial homology of the projektive Plane $\mathbb{R}P^2$, first with $\mathbb{K} = \mathbb{Z}$ and then with $\mathbb{K} = \mathbb{Z}/2$.



Simplicial presentation of a torus, the Klein Bottle and the projective plane.

Exercise 2.5 (Five-Lemma)
Consider the commutative diagrame of $\mathbb{K}$-modules with exact rows:

$$\begin{tikzcd}
A_5 \arrow[r] \arrow[d, "f_5"'] & A_4 \arrow[r] \arrow[d, "f_4"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_1 \arrow[d, "f_1"'] \\
B_5 \arrow[r] & B_4 \arrow[r] & B_3 \arrow[r] & B_2 \arrow[r] & A_1
\end{tikzcd}$$

Prove:
(a) If f_2 and f_4 are epimorphisms and f_1 a monomorphism, then f_3 is an epimorphism.
(b) If f_2 and f_4 are monomorphisms and f_5 an epimorphism, then f_3 is a monomorphism.

Exercise 2.6* (Grad, rot and div)
(1) We consider for an open subset X of $\mathbb{R}^2$ real vector spaces defined as

$C_0(X) = C^\infty(X)$, the space of all real smooth functions $f(x, y)$ on X,
$C_{-1}(X) = C^\infty(X) \times C^\infty(X)$, to be interpreted as the space of all vector fields $v(x, y) = (v_1(x, y), v_2(x, y))$ on X,
$C_{-2}(X) = C^\infty(X)$, now to be thought as the space of all volume forms on X.
There are linear maps

$$\mathrm{grad}\colon C_0(X) \to C_{-1}(X), \quad \mathrm{grad}(f) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$$

and

$$\mathrm{rot}\colon C_{-1}(X) \to C_{-2}(X), \quad \mathrm{rot}(v_1, v_2) = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}.$$

Show with the help of formulas from vector calculus:

$$\mathrm{rot} \circ \mathrm{grad} = 0.$$

Interpret this as a boundary operator, define chains, cycles and boundaries and homology groups $\mathcal{H}_0(X)$ and $\mathcal{H}_{-1}(X)$. For arbitrary X, find the dimension of $\mathcal{H}_0(X)$, i.e., the dimension of the kernel of grad. When X is not simply-connected, give an example of a vector field with zero rotation which is not the gradient of a function, thus showing that $\mathcal{H}_{-1}(X)$ is non-zero.

(2) Now consider an open subset X of $\mathbb{R}^3$. Set up the vector spaces of 0-forms (funtions), of 1-forms (vector fields), of 2-forms (again vector fields) and 3-forms (volume forms) and define the linear map grad, rot and now also the divergence div of a vector field. Show that

$$\mathrm{rot} \circ \mathrm{grad} = 0 \quad \text{and} \quad \mathrm{div} \circ \mathrm{rot} = 0.$$

So we have again a chain complex. Can you compute some homology groups ?

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$g\,dx + h\,dy$ with $\partial g/\partial y = \partial h/\partial x$ (in other words, an exact differential); an expression $k\,dx\,dy$. If the domain D of definition is, say, the interior of the square we can write the function k as $\partial h/\partial x$ for a suitable h, while any exact differential can be expressed (by suitable integration) as the differential of a function f. Hence, for this D the only homology classes are those yielded by the constant functions, and $H(C)$ is the additive group of real numbers. The same conclusion holds if D is the interior of a circle, but fails if D is, say, the interior of a circle with the origin deleted. In this latter case an exact differential need not be the differential of a function f. For example $(-y\,dx + x\,dy)/(x^2 + y^2)$ is not such.

.... continued, ibidem, p.38.