Aufgaben zur Topologie I

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Blatt 1 due: 15.10.2019

Henry Poincaré (1854 - 1912) introduced the fundamental group and the homology groups.

Exercise 1.1 (Homotopy)

- Recall the definition of a homotopy for maps $f\colon X \to Y$, and the notion of a relative homotopy for maps $f\colon (X, A) \to (Y, B)$, in particular the notion of a based homotopy.
- Recall the notion of a homotopy class.
- Show that for self-maps of $\mathbb{S}^1$, free and based homotopy classes agree.

Exercise 1.2 (Degree)
Recall the definition of degree $\mathrm{grad}(f)$ for self-maps of $\mathbb{S}^1$ and recall the basic properties, using the isomorphism $\mathrm{grad}\colon \pi_1(\mathbb{S}^1, 1) \longrightarrow \mathbb{Z}$.

Exercise 1.3 (Fundamental group of a retract)
Let (X, x_0) be a based retract of (Y, y_0), i.e., there is an *inclusion* $\iota\colon (X, x_0) \to (Y, y_0)$ and a *retraction* $R\colon (Y, y_0) \to (X, x_0)$ with $R \circ \iota = \mathrm{id}_X$.

(a) Show that there is a short exact sequence of groups (also called an extension of groups)

$$1 \longrightarrow N \longrightarrow \pi_1(Y, y_0) \xrightarrow{R_*} \pi_1(X, x_0) \longrightarrow 1\,,$$

where $N := \ker(R_*)$ is the kernel of R_* and thus a normal subgroup.

(b) This sequence splits, i.e. there is a section $s\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ with $R_* \circ s = \mathrm{id}$. Consequently, $\pi_1(Y, y_0)$ is a semi-direct product $N \rtimes \pi_1(X, x_0)$.

(c) For two spaces A and B, can $A \vee B \subset A \times B$ be a retract ? Derive a necessary condition and use it to find a counter-example.

Exercise 1.4 (On what does the fundamental group depend ?)
a) Formulate correctly and prove: $\pi_1(X, x_0)$ depends only on the path-component of x_0.
b) Formulate correctly and prove: If X is path-connected, then $\pi_1(X, x_0)$ is independent of x_0.

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https://archive.org/details/uvresdehenripoin06poin/page/206

Under this address one finds Poincaré's article *Analysis Situs* (1895), where he defines on page 206 the notion of homology. In the next paragraph he defines the Betti numbers.

Exercise 1.5 (Fundamental groups of complements)
Regard $\mathbb{R}^m$ as a subspace of $\mathbb{R}^n$ by setting the last $n - m$ coordinates equal to zero; and likewise, regard $\mathbb{S}^m$ as a subspace of $\mathbb{S}^n$. Assume $m < n - 1$.

(a) Compute $\pi_1(\mathbb{R}^n \setminus \mathbb{R}^m)$.

(b) Compute $\pi_1(\mathbb{S}^n \setminus \mathbb{S}^m)$.

Poincaré als junger Mann.

Exercise 1.6* (2-dimensional intermediate value theorem)
Let $f\colon \mathbb{D}^2 \to \mathbb{R}^2$ be a smooth function and $P \in \mathbb{R}^2$ a point not in the image $f(\partial\mathbb{D}^2) = f(\mathbb{S}^1)$ of the boundary. Recall the Umlauf number $\mathrm{Uml}(f, P)$ of f with respect to P. Show that f must assume the value P, if $\mathrm{Uml}(f, P) \neq 0$.