Themen für Bachelorarbeiten

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1 Cubical Homology

The singular chain complex $S_\bullet(X)$ of a space X has as a basis in degree n all continouous map $a\colon \Delta^n \to X$, the so-called singular simplices. If we replace these by singular cubes $c\colon I^n \to X$, where $I^n = [0,1]^n$, we get the cubical chains; we have seen in Exercises 4.4 and 5.6 that we need to divide out the degenerate cubes; in the end we have the cubical complex $K_\bullet(X)$ with the appropriate differential $\partial\colon K_n(X) \to K_{n-1}(X)$. Thus we can define cubical homology groups $HK_n(X) := H_n(K_\bullet(X), \partial)$ as the homology of the chain complex $K_\bullet(X)$.

The goal of this bachelor thesis is the **Theorem**: *$HK_*(X)$ is a homology theory. Furthermore, there is a natural equivalence $\Psi\colon H_*(X) \to HK_*(X)$.* One needs to prove the Eilenberg-Steenrod axioms and establish such an equivalence. A prominent place in the literaure is [Serre, Chap. II.1]; for a textbook see [Toe, 7.5].

References

[Serre] J.-P. Serre: *Homologie singuliere des espaces fibres. Application.* Ann. Math. 54 (1951), 425-505.

[Toe] F. Toenniessen: *Topologie*. Springer Spektrum (2017).

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2 Transfer

For a continous map $f\colon X \to Y$ we have we have in each degree k an induced map in homology $f_*\colon H_k(X) \to H_k(Y)$. But for finite coverings $\xi\colon \tilde{(X)} \to X$ with n sheets we have in addition a homomorphism $T_\xi\colon H_k(X) \to H_k(\tilde{X})$, defined as follows: to any singular simplex $a\colon \Delta^k \to X$ in $S_k(X)$ we associate $\tilde{a} := \tilde{a}_1 + \ldots + \tilde{a}_n$ in $S_k(\tilde{X})$, where the $\tilde{a}_i$ are the lifts of a, so $a = \xi \circ \tilde{a}_i$. There are exactly n such lifts. This association $a \mapsto \tilde{a}$ is a chain map $S_k(X) \to S_k(\tilde{X})$ and we call the induced homomorphism $T_\xi\colon H_k(X) \to H_k(\tilde{X})$ the *transfer homomorphism* of ξ. It is natural with respect to maps of coverings; and it has the important property

$$\xi_* \circ T_\xi = n\colon H_k(X) \longrightarrow H_k(X)\,,$$

the multiplication with $n \in \mathbb{Z}$ on $H_k(X)$. This has important consequences, for example: If $\tilde{X}$ is acyclic, then the reduced homology of X is all torsion of order prime to n.

The goal of the bachelor thesis is to develop this theory in homology and cohomology and apply it to several examples. A good reference is [Ha, 3.G].

References

[Ha] A. Hatcher: *Algebraic Topology.* Cambridge University Press.

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3 Generalized Borsuk-Ulam Theorem

The classical Borsuk-Ulam Theorem says, that for any map $f\colon \mathbb{S}^n \to \mathbb{R}^n$ there must be a point $x \in \mathbb{S}^n$ with $f(x) = f(-x)$. There are certain generalisations for manifolds M instead of $\mathbb{S}^n$ with an involution instead of the antipodal map of the sphere.

The goal of this bachelor thesis is to work out the first sections of [G-G].

References

[G-G] D. Lima Gonsalves & J. Guaschi: *The Borsuk-Ulam Theorem for maps into a surface.* Top. Appl. 157 (2010), 1742-1759.

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4 Combinatorics of Steenrod Squares

In the cohomology theory $H^*(X;\mathbb{Z}/2)$ with mod 2 coefficients there are important operations

$$Sq^i\colon H^n(X;\mathbb{Z}/2) \longrightarrow H^{n+i}(X;\mathbb{Z}/2)\,,$$

called the *Steenrod operation*. They are natural homomorphism and satisfy certain axioms, e.g. $Sq^i(\alpha) = \alpha \cdot \alpha$ for $i = |\alpha|$; the multiplication in this formula is the cup-product, which turns $\mathbb{H}(X) := \bigoplus_{k\geq 0} H^k(X;\mathbb{Z}/2)$ into a graded, associative and graded-commutative ring with unit. [We will learn this cup product in the summer term.] The definition of the Steenrod squares is somewhat involved; there are homological constructions as in [Bre, VI.15 + 16] and homotopical constructions as in [Ha, 4.L].

The goal of the bachelor thesis is to go back to the original article of Steenrod [Ste] and understand the combinatorial background in Part I. Then one follows the homological path and constructs these operations.

References

[Bre] G. Bredon: *Topology and Geometry.* Springer Verlag.

[Ha] A. Hatcher: *Algebraic Topology.* Cambridge University Press.

[Ste] N. Steenrod: *Products of cocycles and extensions of mappings.* Ann. Math. 48 (1947), 290-320.

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5 Homology with local coefficients

In the definition of the homology $H_k(X)$ the coefficients are in a fixed ring, say the ring $\mathbb{Z}$. There are geometric applications which suggest that it might be a good idea to let the coefficients vary, i.e., the coefficient of a singular simplex $a\colon \Delta^k \to X$ shall be in a ring $\mathcal{A}_x$ associated to the point $x \in X$, where $x = a(\frac{1}{n+1}, \dots, \frac{1}{n+1})$ is the image under a of the barycenter of the simplex Δ^n. We assume that each $\mathcal{A}_x$ is isomorphic to $\mathbb{Z}$, but it does not make sense to say that all $\mathcal{A}_x$ are equal to $\mathbb{Z}$. And we need to move coefficients from $\mathcal{A}_x$ to $\mathcal{A}_y$ along a path w with $w(0) = x$ and $w(1) = y$.

In fashionable words: $\mathcal{A}$ is a functor from the fundamental gruppoid $\Pi_0(X)$ to the the categorie of commutative rings. The definition of the differential $\partial\colon S_n(X;\mathcal{A}) \to S_{n-1}(X;\mathcal{A})$ is a bit intricate. We call the homology of this sinular chain complex the *singular homology of X with coefficients in $\mathcal{A}$.*

The goal of this bachelor thesis is to develop this theory and give many applications. A good introduction is [Ha, 3.H]; the connection to sheafs is treated in [Spa, p. 360]. For the important example of the orientation bundle, see [Bre, VI.7].

References

[Bre] G. Bredon: *Topology and Geometry.* Springer Verlag (1993).

[Ha] A. Hatcher: *Algebraic Topology.* Cambridge University Press.

[Spa] E. Spanier: *Algebraic Topology.* McGraw Hill.

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6 2-fold Coverings and H^1

Let $\mathrm{Cov}_2(X)$ denote the set of isomorphism classes of 2-fold coverings of X. From the theory of covering spaces we know a bijection between $\mathrm{Cov}_2(X)$ and the conjugacy classes of index-2 subgroups of $\pi_1(X, x_0)$. The latter subgroups in turn are exactly the kernels of homomorphisms $\pi_1(X, x_0) \to \mathbb{Z}/2$.

Furthermore, if $f\colon X \to \mathbb{R}\mathrm{P}^\infty$, the pull-back of the universal 2-fold-covering $\mathbb{S}^\infty \to \mathbb{R}\mathrm{P}^\infty$ along f gives a 2-fold covering of X. With the same f we can pull-back the generator $\alpha \in H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2)$ to $f^*(\alpha) \in H^1(X;\mathbb{Z}/2)$. (There is also an obvious connection between 2-fold coverings and real line bundles over X by taking the 0-dimensional sphere in the line over ich point, which makes all this even more interesting.)

The goal of this bachelor thesis is to show: There is a bijection between $\mathrm{Cov}_2(X)$, the first cohomology group $H^1(X;\mathbb{Z}/2)$ and the set of homotopy classes $[X, \mathbb{R}\mathrm{P}^\infty]$. A good reference is [Hau, 4.3].

References

[Hau] J-Cl. Hausmann: *Mod Two Homology and Cohomology.* Springer (2014).

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7 Lefschetz Theory

Assume X has finite type homology over a field, i.e., all homology groups $H_*(X;\mathbb{F})$ with coefficients in a field $\mathbb{F}$ are finite-dimensional and almost all are trivial. Then one can define to any self-map $f\colon X \to X$ a so-called *Lefschetz number*

$$L(f) := \sum_i (-1)^i \ \mathrm{Trace}(H_i(f)\colon H_i(X;\mathbb{F}) \to H_i(X;\mathbb{F}))\,.$$

This is a generalisation of the Euler number, since $\chi(X) = L(\mathrm{id}_X)$. The famous Lefschetz Fixed Theorem says: If X is a compact polyhedron and $L(f) \neq 0$, then f has a fixed point.
The theory needs the simplicial approximaion theorem as a tool. (There are interesting applications for coincidence questions, for which Poincare duality will be needed; see [Bre, VI.14].)

The goal of the bachelor thesis is to develop the theory, go over many applications and also try the Lefschetz coincidence theory.

References

[Bre] G. Bredon: *Topology and Geometry.* Springer (1993).

8 Euler number of a map

Assume X has finite type homology over $\mathbb{Q}$, all homology groups $H_*(X;\mathbb{Q})$ with coefficients in $\mathbb{Q}$ are finite-dimensional and almost all are trivial. Then one can define to any self-map $f\colon X \to X$ a so-called *Euler number*

$$\chi(f) := \sum_i (-1)^i \ \dim(H_i(X;\mathbb{Q})/K_i)\,.$$

where K_i is the union of the ascending chain of subvector spaces $\ker(f_*^m)$ given by the kernels of the homomorphisms induced by the iteration of $f_*^m\colon H_i(X;\mathbb{Q}) \to H_i(X,\mathbb{Q})$ in degree $i \geq 0$. Clearly, this is a generalization of the Euler number of a space, since $\chi(X) = \chi(\mathrm{id}_X)$. In the same way as the Lefschetz number of a map detects fixed points, this Euler number detercs periodic points.

The goal of this bachelor thesis is to prove: *For X a finite connected polyhedron, any self-map $f\colon X \to X$ has a periodic point, if $\chi(f) \neq 0$.*
The proof in [G-D, III.9] and the applications should be worked out in detail.

References

[G-D] A. Granas & J. Dugundji: *Fixed Point Theory.* Springer (2003).

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