Exercises for Algebraic Topology II

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Blatt 12 due by: 16.07.2018

R. Thom, G. Reeb and J.-P. Serre, 1949 in Oberwolfach (from left to right)

Exercise 12.1 (Euler characteristic of a spectral sequence)

For a chain complex $C_\bullet$ of abelian groups, which is of finite type, one defines the Euler characteristic as

$$\chi(C_\bullet) = \sum_n (-1)^n \operatorname{rank}(C_n) = \sum_n (-1)^n \dim_{\mathbb{Q}}(C_n \otimes_{\mathbb{Z}} \mathbb{Q}).$$

We know that $\chi(C_\bullet) = \chi(H(C_\bullet))$.

Now for a spectral sequence $E = (E^r_{p,q})$ of abelian groups define the notion of finite type. Define for each page E^r an Euler characteristic $\chi(E^r)$. Show that E^{r+1} is of finite type, if E^r is. And conclude $\chi(E^r) = \chi(E^{r+1})$. We call this constant number the *Euler charcteristic* of E, denoted by $\chi(E)$.

Prove that E does converge, if it is of finite type. Say it converges to the graded and filtered abelian group H. Show that H is also of finite type and that $\chi(H) = \chi(E)$.

Exercise 12.2 (Poincare polynomial of a spectral sequence)

Recall the Poincare polynomial $P_t(C_\bullet) = \sum_n (-1)^n \operatorname{rank}(C_n)\, t^n$ of a chain complex or graded module of abelian groups. Under what conditions can one define it? Is there a relation to the homology $H(C_\bullet)$? — Now define a Poincare polynomial for a spectral sequnce and show the obvious.

Jean-Pierre Serre

Exercise 12.3 (Serre classes I)

A *Serre class* of abelian groups is a non-empty class $\mathcal{S}$ of abelian groups with the property: if in an exact sequence $A \to B \to C$ we have $A, C \in \mathcal{S}$, then also $B \in \mathcal{S}$.

(a) Show for some of the following examples that they are Serre classes: all abelian groups, all trivial groups, all finite groups, all finitely generated groups, all torion groups, all p-torion groups (p any prime), all p-local groups (i.e. torsion groups with all orders of elements being prime to p).

(b) One calls a homomorphism $\phi\colon A \to B$ a $\mathcal{S}$-monomorphism resp. a $\mathcal{S}$-epimorphism, if the kernel resp. the cokernel of ϕ is in $\mathcal{S}$. Now it is clear what we mean by a $\mathcal{S}$-isomorphism. Show that $\mathcal{S}$-ismorphic is an equivalence relation.

Exercise 12.4 (Serre classes II)

Let $E = (E^r_{p,q})$ be a spectral sequence of abelian groups and let $\mathcal{S}$ be a Serre class. Assume E converges to the filtered and graded abelian group H.

(a) If, for a fixed r, all $E^r_{p,q}$ are in $\mathcal{S}$, then all $E^{r+1}_{p,q}$ are in $\mathcal{S}$.

(b) In this case, also all H_n are in $\mathcal{S}$.

Und nochmal Serre.