Exercises for Algebraic Topology II

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Exercise 9.1 (Associated principal bundles and the Borel construction)

Let $\xi\colon E \to B$ be a fibre bundle with fibre F and structure group G, and $\zeta\colon P \to B$ be a principal G-bundle over B. We can form the *Borel construction* $E := P \times_G F$ with its projection to B. We denote this fibre bundle by $\xi = \zeta \times_G F$. Vice versa, given an (F, G)-fibre bundle $\xi\colon E \to B$ we denote by $\mathrm{prin}(\xi)$ the associated principal bundle with total space $P = \mathrm{bunmap}(\tau_F, \xi)$, the space of bundle maps from the trivial (F, G)-bundle $\tau_F\colon F \to *$ over a point to the bundle ξ.

Show some of the following formulae for theses constructions:

(1) $\mathrm{prin}(\xi) \times_G F \;\cong\; \xi$

(2) $\mathrm{prin}(\zeta \times_G F) \;\cong\; \zeta$, if G acts faithfully on F.

(3) $\zeta \times_G G \;\cong\; \zeta$

(4) $\mathrm{prin}(\zeta) \;\cong\; \zeta$

(5) $\mathrm{prin}(f^*(\xi)) \;\cong\; f^*(\mathrm{prin}(\xi))$ for any $f\colon B' \to B$.

(6) $f^*(\zeta) \times_G F \;\cong\; f^*(\zeta \times_G F)$ for any $f\colon B' \to B$.

Armand Borel (1923 — 2003)

Exercise 9.2 (Classifying spaces for subgroups)

Let $H \leq G$ be a subgroup of a topological group, and

(a) If $\zeta\colon P \to B$ is a principal G-bundle, then $\zeta_H\colon P \to B' := P/H$ is a principal H-bundle.

(b) If $\omega_G\colon EG \to BG$ is a universal G-bundle, then $\omega_H\colon EG \to BH := EG/H$ is a universal H-bundle.

(c) The induced map $\xi\colon BH = EG/H \to BG = EG/G$ is a fibre bundle with fibre the homogeneous space $F := G/H$ and structure group G.

Exercise 9.3 (Fixed points and sections)

Let $\zeta\colon P \to B$ be a principal G-bundle. Let F be a G-space and $A := \mathrm{Fix}(F)$ the set of G-fixed points in F. Show that that for any fixed point $a \in A$ there is a section $s_a\colon B \to E$ in the fibre bundle $\xi := \zeta \times_g F\colon E = P \times_G F \to B$. More generally, ξ contains the trivial bundle $\xi_A := P \times_G A \cong P/G \times A \to B$.
(Hint: The word 'contains' means as a subbundle. What is a subbundle ? Cf. Exercise 9.4.)

Exercise 9.4 (Fibre exchange)

Let $\xi\colon E \to B$ be a fibre bundle with fibre F and structure group G. Let F' be another G-space and $\phi\colon F \to F'$ be a G-equivariant map. We consider the new fibre bundle ξ' over B with total space $E' := \mathrm{prin}(\xi) \times_G F$, obvious bundle projection ξ and the new fibre F'.
Show that there is a generalized bundle map $(\tilde{f}, \mathrm{id}_B)\colon \xi \to \xi'$, i.e., a commutative square

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\xi"'] & E' \arrow[d, "\xi'"] \\ B \arrow[r, "\mathrm{id}"'] & B \end{tikzcd}$$

such that the restriction of $\tilde{f}$ to any fibre is conjugate to ϕ; more precisely, the composition $F \to E_b \to E'_b \to F$, where the middle map is the restriction $\tilde{f}|\colon E_b \to E'_b$, the first map resp. the third map is an identication of the actual fibre with F resp. F' via some bundle coordinates, is $g' \circ \phi \circ g$ for some $g, g' \in G$.
(Hint: The phrasing of the exercise is essentially the definition of generalized bundle map.)

Exercise 9.5* (Gauge groups and adjoint representation)

Let $\zeta\colon P \to B$ be a principal G-bundle for a Lie group G. The *gauge group* $\mathcal{G}(\zeta)$ of this bundle is the group of bundle isomorphisms $(\tilde{f}, \mathrm{id}_B)\colon \zeta \to \zeta$. It is a topological group (with one of the obvious topologies); like G, it acts on P.

(a) $\mathcal{G}(\zeta)$ contains G; is it larger than G ?

The conjugation $G \times G \to G, (g, \gamma) \mapsto g\gamma g^{-1}$ can be regarded as a left action of G on itself. Thus we can define the (F, G)-bundle $\zeta \times_G G$ with structure group G and fibre $F = G$, but with the left conjugaion action, not the left translation action. The bundle is called the *adjoint bundle* and denoted by $\mathrm{Ad}(\zeta)$.

(b) Show that its space of sections $\mathrm{Sect}(\mathrm{Ad}(\zeta))$ forms a group (under fibrewise multiplication).

(c) Show that there is an isomorphism of group $\mathcal{G}(\zeta) \cong \mathrm{Sect}(\mathrm{Ad}(\zeta))$.