Exercises for Algebraic Topology II

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Blatt 8 due by: 18.06.2018

Exercise 8.1 (Pull-backs of bundles)

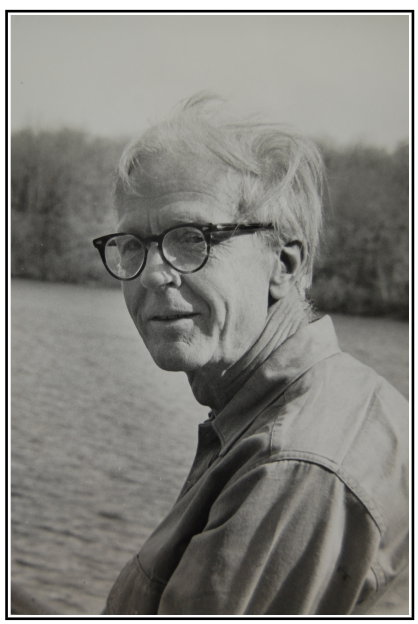
Show that the pull-back of an (F, G)-bundle is an (F, G)-bundle.
Show that the pull-back of a principal G-bundle is a principal G-bundle.

Exercise 8.2 (Stiefel bundles over Graßmann manifolds)

Show that the projections from Stiefel manifolds to Graßmann manifolds

$$V_n(\mathbb{R}^k) \longrightarrow \mathrm{Gr}_n(\mathbb{R}^k) \qquad \text{resp.} \qquad V_n(\mathbb{C}^k) \longrightarrow \mathrm{Gr}_n(\mathbb{C}^k),$$

sending an orthogonal resp. unitary n-frame to its linear span, are principal G-bundles for $G = O(n)$ resp. for $G = U(n)$, where $1 \leq n \leq k \leq \infty$.



Hassler Whitney (1907 - 1989)

Exercise 8.3 (Hopf-Whitney Classification Theorem of maps to $\mathbb{S}^n$)

We consider for any based space X the based homotopy classes of maps into a sphere $\mathbb{S}^n$ with $n \geq 1$ and define the natural transformation

$$[X, \mathbb{S}^n] \longrightarrow H^n(X; \mathbb{Z}), \qquad [f] \mapsto \Phi([f]) := f^*(\omega_n),$$

where $\omega_n \in H^n(\mathbb{S}^n)$ is a generator. Prove that Φ is surjective for any connected n-dimensional CW complex X.

Remark: Φ is even bijective (for n-dimensional CW complexes); this is the full Hopf-Whitney classification theorem. To prove the injectivity one needs a bit of obstruction theory. See G. W. Whitehead: *Elements of Homotopy Theory*, p. 244.

Example: If X is a compact, connected, oriented and triangulated m-manifold, then $\Phi([f])$ evaluated at the fundamental class u of X (i.e., Kronecker product with u) is what we called earlier the degree of f.

Exercise 8.4 (Classifying map for universal coverings)
Show that for any connected space X with basepoint x_0 there is a map $X \to B\pi_1(X, x_0)$, which induces an isomorphism between fundamental groups.

Exercise 8.5* (Infinity symmetric products and Poincare-Lefschetz duality)
Let M be an m-manifold and M_0 a submanifold of arbitrary dimension. For any W is an m-manifold containing M we consider the tangent bundle $\tau\colon T(W) \to W$ and its fibrewise one-point-compactification $\sigma\colon \dot{T}(W) \to W$. Next we denote by $\pi\colon \mathrm{SP}(\sigma) \to W$ the fibrewise infinite symmetric product of $\dot{T}(W)$; it is a fibre bundle with fibre $\mathrm{SP}(\mathbb{S}^m, \infty)$ an Eilenberg-MacLane space and structure group $\mathrm{GL}_m(\mathbb{R})$. The action of G has, for each $z \in W$, only one fixed point on each tangent space $\mathbb{R}^m = T_z(M)$, namely the origin 0, but has two fixed points of the one-point-compactification $\mathbb{S}^m = \dot{T}_z(W)$, namely 0 and ∞. Thus we have two section for the bundle σ and also two sections for the bundle π, the latter we denote by s_0, s_∞. By $\mathrm{Sect}(W - M_0, W - M; \pi)$ we denote the space of sections of π, which are defined on $W - M_0$ and agree with s_∞ on $W - M$.

(1) Find a 'scanning map'
$$\gamma\colon \mathrm{SP}(M, M_0) \longrightarrow \mathrm{Sect}(W - M_0, W - M; \pi).$$

(2) Prove that γ is a weak homotopy equivalence, if M is compact and the pair (M, M_0) is connected.

Now we use the following fact: $\pi\colon \mathrm{SP}(\sigma) \to W$ *is fibre-homotopy trivial iff* W *is orientable.* And we continue to ask:

(3) Why is this exercise called 'Infinite symmetric products and Poincare-Lefschetz duality' ?

(4)* And what are the homotopy groups of the right-hand side, if W is not orientable ?