Exercises for Algebraic Topology II

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Summer Term 2018

Blatt 5 due by: 28.05.2018

Graeme Segal, ca. 2009

Exercise 5.1 (Labeled configuration spaces, H-spaces and the little cube operad)

Let M be an m-manifold, and M_0 an m-submanifold; let X be a connected space with basepoint x_0.

(1) $L^k = C(M \times [0,1]^k, M_0 \times [0,1]^k; X)$ is a k-fold loop space. (Even if X is not connected, L^1 is an H-space.)

(2) The operad $\mathcal{C}^k = (C^n(\mathbb{R}^k)_{n\geq 0})$ of little k-cubes operates on L^k.

(3) The H-space $H = C(\partial M \times [0,1], (\partial M \cap M_0) \times [0,1]; X)$ acts on $C(M, M_0; X)$.

(4)* The spaces $S^k = C(M, M_0; \Sigma^k X)$ and $T^k = C(M \times [0,1]^k, M \times \partial([0,1]^k) \cup M_0 \times [0,1]^k; X)$ have (by the Approximation Theorem) the same (weak) homotopy type. Can you find a map

$$S^k = C(M, M_0; \Sigma^k X) \longrightarrow T^k = C(M \times [0,1]^k, M \times \partial([0,1]^k) \cup M_0 \times [0,1]^k; X),$$

which is a (weak) homotopy equivalence ? (Here we assume that M is compact.)

Exercise 5.2 (Stable summands in mapping spaces)

Let K be a finite complex, and $K_0 \subset K$ a subcomplex, X a connected and based space. Show that $K/K_0 \wedge X$ splits off from the mapping space $\mathrm{map}(K, K_0; \Sigma^m X)$ after m suspensions, when m is greater or equal to the embedding dimension of K.

Exercise 5.3 (Symmetric product of a circle)

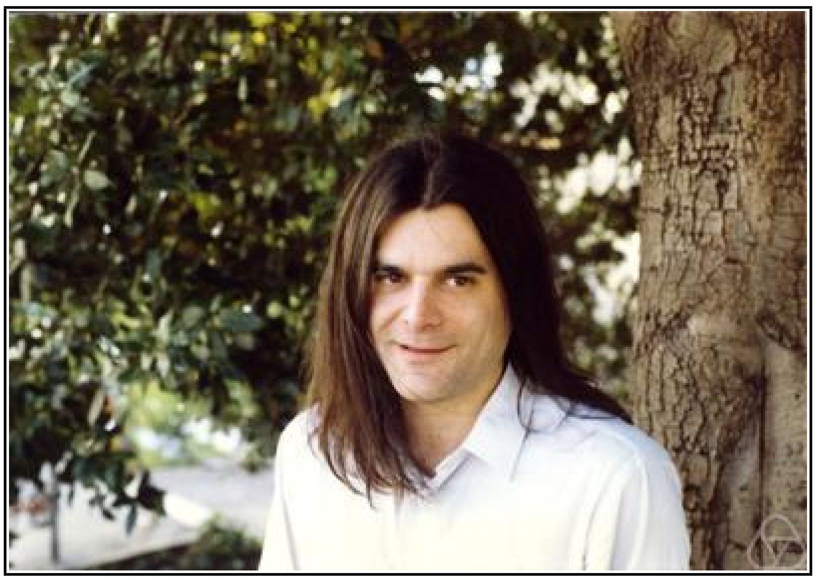
Since $\mathbb{S}^1$ is an abelian Lie group, there is a continous function $\mu\colon \mathrm{SP}_n(\mathbb{S}^1) \longrightarrow \mathbb{S}^1$, which simply multiplies all elements of a weighted set ζ, i.e.,

$$\mu(\zeta) = \mu(\sum_i k_i\, z_i) \quad := \quad \prod_i {z_i}^{k_i} \qquad \text{(in the notation with multiplicities)} \tag{1}$$

$$\mu(\zeta) = \mu([z_1, z_2,\ \ldots,\ z_n]) \quad := \quad z_1 z_2\ \cdots\ z_n \qquad \text{(in the notation without multiplicities).} \tag{2}$$

Show that this surjective map is a fibre bundle with fibre a $(n-1)$-simplex Δ^{n-1}. Furthermore, the bundle is trivial for n odd; it is a Möbius band for $n = 2$; and for arbitrary even n it is the (Whitney) sum of a Möbius band and trivial bundle.

Remark: To a unitary matrix A we can associate the 'weighted collection' of its eigenvalues with their multiplicities; this is an element $\mathcal{E}(A) \in \mathrm{SP}_n(\mathbb{S}^1)$. The map $\mathcal{E}\colon U(n) \to \mathrm{SP}_n(\mathbb{S}^1)$ is continuous and invariant under conjugation. One can show: $U(n)/\text{conjugation} \cong \mathrm{SP}_n(\mathbb{S}^1)$. The composition $\mu \circ \mathcal{E}\colon U(n) \to \mathbb{S}^1$ is obviously the determinant.



Graeme Segal, ca. 1982

Exercise 5.4 (Symmetric product of a bouquet)

For two connected and based spaces X and Y show that $\mathrm{SP}(X \vee Y) \cong \mathrm{SP}(X) \times \mathrm{SP}(Y)$.

Application: The map 'multiplication by n', denoted by $M_n\colon\ \mathrm{SP}(X) \to \mathrm{SP}(X)$ and given by $M_n(\zeta) := M_n(\sum k_i\, x_i) := n\,(\sum_i k_i\, x_i) = \sum_i nk_i\, x_i$, induces on $\pi_q(\mathrm{SP}(X))$ the multiplication by n. (Note: Since $\mathrm{SP}(X)$ is an H-space, all homotopy groups are abelian.)

Exercise 5.5* (Symmetric products, coverings, and transfer)

Let $p\colon \tilde{X} \to X$ be an n-fold covering. We define for p first a *pre-transfer* $X \to \mathrm{SP}_n(\tilde{X})$ by taking the inverse image of points, regarding it as an unorderd set with n elements. We extend this easily to a map $\mathrm{SP}_k(X) \to \mathrm{SP}_{nk}(\tilde{X})$ and further to a *tranfer* $\tau_p\colon\ \mathrm{SP}(X, x_0) \to \mathrm{SP}(\tilde{X}, p^{-1}(x_0))$

(1) What can we say about the composition $\mathrm{SP}(p) \circ \tau_p\colon\ \mathrm{SP}(X, x_0) \to \mathrm{SP}(X, x_0)$?

(2) Conclude, that the rational homology $H_q(X;\mathbb{Q})$ is a direct summand in the rational homology $H_q(\tilde{X};\mathbb{Q})$. Same for rational cohomology, or for (co)homology with coefficients in a field of characteristic prime to n.

(3) The integral homology $H_*(G;\mathbb{Z})$ of a discrete group G is isomorphic to the integral homology of its classifying space BG, and BG can be obtained as E/G for any contractible space E with a free G-action. So for a finite G we have a finite covering $p\colon E \to E/G = BG$. Show that the rational homology $H_q(G;\mathbb{Q})$ of any finite group is trivial for $q > 0$.

(4*) Can one define a transfer also for branched coverings ? — The notion of a branched covering is not easily defined; here we think of two kinds of branchings: (a) branched coverings of two surfaces $p\colon F' \longrightarrow F$, and (b) quotients $p\colon E \longrightarrow B = E/G$, where a finite group acts properly-discontinuously on E, but perhaps not freely.

A.1.5 Example. Suppose that $B = \mathbb{R}^2$ and E is the plane with a cut along the interval $0 < x < 1$, $y = 0$ without the lower boundary, that is, without the boundary of the region $y < 0$. In other words, E is the result of taking the upper half-plane $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$ and the part of the lower half-plane $\mathbb{R}^2_- = \{(x,y) \in \mathbb{R}^2 \mid y \leq 0\}$ from which one takes away the said interval, and identifying the half-lines $\{(x,0) \mid x \leq 0\} \cup \{(x,0) \mid x \geq 1\}$ of both via the identity. Let $p : E \longrightarrow B$ be the natural projection (see Figure A.1).

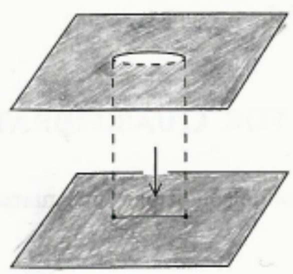


Figure A.1

The open half-planes $U = \{(x,y) \mid x > 0\}$ and $V = \{(x,y) \mid x < 1\}$ are distinguished, since the groups $\pi_n(p^{-1}(U), p^{-1}(b))$, $\pi_n(U)$, $\pi_n(p^{-1}(V), p^{-1}(b))$, and $\pi_n(V)$ are all trivial. Moreover, they cover B. If p were a quasifibration, then we would have an isomorphism $p_* : \pi_n(E) \cong \pi_n(B)$, since all of the fibers are points. However, the group $\pi_n(B)$ is trivial, while $\pi_n(E)$ is infinite cyclic because E has the homotopy type of the circle $\mathbb{S}^1$ (see 4.5.13).

The previous example shows also that a subset of a distinguished set is not necessarily distinguished. The half-plane U is distinguished, but the strip $0 < x < 1$ is not (otherwise, the whole plane would be distinguished by Theorem A.1.2). In particular, this proves that a map $B' \longrightarrow B$ into the base space of a quasifibration $E \longrightarrow B$ does not in general induce a quasifibration $E' \longrightarrow B'$.

Pull-backs of quasifibrations need not be quasifibrations.
M. Aguilar, S. Gitler, C. Prieto: **Algebraic Topology from a Homotopical Viewpoint**. Springer Verlag (2002), p. 422.