Exercises for Algebraic Topology II

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Exercise 1.1 (Components of an H-space)

Let M be an H-space and 1 its h-neutral element.

(1) If M has an homotopy inverse, all its path components are homotopy-equivalent.

(2) If M_0 denotes the path component of 1, then M_0 is also an H-space.

Exercise 1.2 (Some properties of H-spaces)

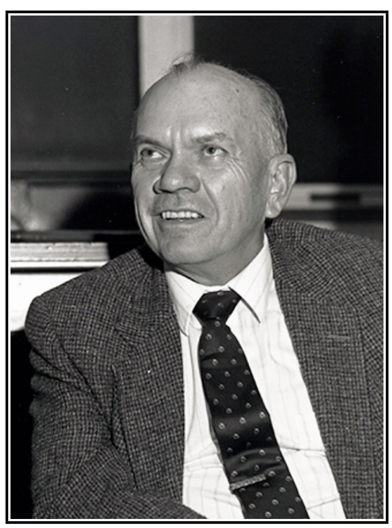
Let M be an H-space and its h-neutral element be 1.

(1) The set $\pi_0(M)$ of its path components is a (discrete) magma with identity; it is commutative, if M is h-commutative; it is associative, if M is h-associative; it is a group, if M has an h-inverse.

(2) $\pi_1(X, 1)$ is abelian.

(3) If M' is another H-space, then $M \times M'$ is an H-space.

(4) For any space X the mapping space $\mathrm{map}(X, M)$ is an H-space.



John C. Moore (1923 — 2016)

Exercise 1.3 (Moore loop space)

Show that the Moore loop space $M(X, x_0)$ and the ordinary loop space $\Omega(X, x_0)$ of a connected and well-based space X with basepoint x_0 are homotopy equivalent.

Exercise 1.4 (James model $J(X)$)

For a space X with basepoint $*$ we denote by $J(X)$ the free topological monoid generated by the points of X, with $*$ as neutral element. In other words, let $X^{(\infty)}$ be the the weak infinite product, consisting of all 'infinite words' $w = (x_1, \dots,)$ with almost all $x_i = *$. We consider the equivalence relation

$$(x_1, \dots, x_i, \dots, x_n, *, \dots) \sim (x_1, \dots, \widehat{x_i}, \dots, x_n, x_i, *, \dots), \text{ if } x_i = *$$

and define $J(X)$ to be the quotient of $X^{(\infty)}$ modulo this equivalence relation. The equivalence class of $(x_1, \dots, x_n, *, \dots)$ can be imagined as a (finite) word $w = x_1\, x_2 \dots x_n$ in the alphabet X.
By $J_n(X)$ we denote the subquotient of X^n modulo this equivalence relation (restricted to X^n). Obviously $J_1(X) = X$. We have inclusions $J_n(X) \hookrightarrow J_{n+1}(X)$ and $J(X) = \lim J_n(X)$ is the direct limit. The concatenation of words

$$w \cdot v = (x_1, \dots, x_n, *, \dots) \cdot (y_1, \dots, y_m, *, \dots) := (x_1, \dots, x_n, y_1, \dots, y_m, *, \dots)$$

make $J(X)$ into a strictly associative topological monoid with strict neutral element $\mathbf{1}$ represented by the words $(*) \sim (*, *) \sim (*, *, *) \dots$ and so forth. In particular, it is an H-space. This $J(X)$ is called the *reduced product space* or the *James construction* of X.

(1) The quotient $J_n(X)/J_{n-1}(X)$ is homeomorphic to the n-fold smash product $X^{(n)}$.

(2) *Universal property*: If M is a strict associative H-space with strict neutral element 1, the any continuous map $f\colon X \to M$ with $f(*) = 1$ has a unique extention $F = \hat{f}\colon J(X) \to M$, which is a homomorphism of monoids, i.e., $F(wv) = F(w)F(v)$ and $F(1) = 1$.

(3) Let $C(\mathbb{R}; X, x_0)$ be the configuration space of $\mathbb{R}$ with labels in (X, x_0). There is a map $\Phi\colon C(\mathbb{R}; X, x_0) \to J(X)$, namely

$$\Phi(Z) = \Phi(\zeta_1, \zeta_2, \dots, \zeta_n; x_1, x_2, \dots, x_n) = [x_1, x_2, \dots, x_n] = x_1\, x_2 \dots x_n,$$

if $\zeta_1 < \zeta_2 < \dots < \zeta_n$ is the monotone indexed representative of the labeled configuration Z. It preserves the filtrations $C_n(\mathbb{R}; X, x_0)$ resp. $J_n(X)$. And it induces a homotopy equivalence of filtration quotients.

Remark: The map Φ is a homotopy equivalence for any well-pointed space. A homotopy inverse of Φ needs a careful choice of positions $\zeta_1 < \zeta_2 < \dots < \zeta_n$ for the reduced word $w = x_1 x_2 \dots x_n$ (where no x_i is the basepoint), which gives the labels; for this choice one uses a halo function around the basepoint and a strong deformation retraction of a neighbourhood of $*$ onto $*$.

Ioan M. James, *1928

Exercise 1.5* (Homology of $J(X)$)

For a sphere $\mathbb{S}^n$ with $n \geq 1$ we consider the reduced product $J(\mathbb{S}^n)$.

(a) $J(\mathbb{S}^n)$ is $(n-1)$-connected.

(b) $H^q(J(\mathbb{S}^n)) \cong \mathbb{Z}$ if q is divisible by n, and $H^q(J(\mathbb{S}^n)) = 0$ otherwise.

The last statement is not difficult to prove; it follows also from the (much less immediate) splittting $\Sigma J(X) \simeq \Sigma(\bigvee_q X^{(n)})$ of the suspended $J(X)$ into its filtration quotients, for any connected X. This splitting implies that $\tilde{H}^*(J(X)) \cong \bigoplus_n \tilde{H}^*(X^{(n)})$.

Back to $X = \mathbb{S}^n$. Denote a generator in degree q by γ_q, if q is divisible by n.

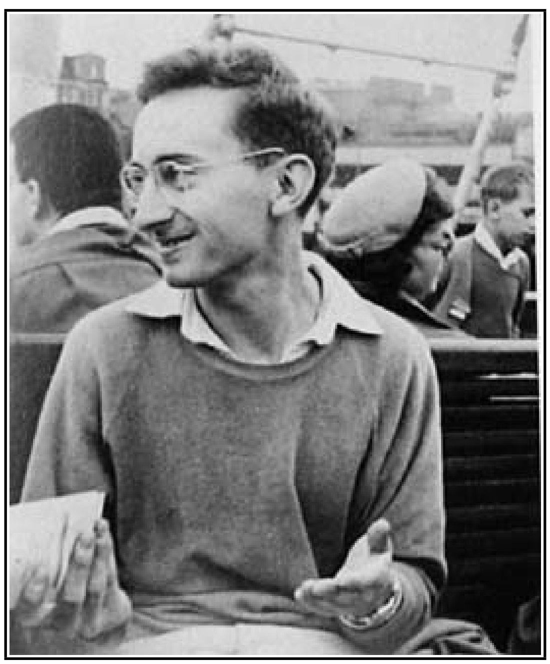
(c) For n even: $\quad \gamma_q^k = k!\ \gamma_{kq}$.

Thus, as a ring, $H^*(J(\mathbb{S}^n)) \cong \Gamma_{\mathbb{Z}}[\gamma_n]$, the divided polynomial algebra generated by $g_k := \gamma_n^k/k!$ inside the polynomial ring $\mathbb{Q}[\gamma_n]$.

(d) For n odd: $\quad \gamma_n^2 = 0, \quad \gamma_{(2l+1)n} = \gamma_n\,\gamma_{2ln}, \quad$ and $\quad \gamma_{2ln}^k = k!\ \gamma_{2lkn}$.

Thus, as a ring, $H^*(J(\mathbb{S}^n)) \cong H^*(\mathbb{S}^n) \otimes H^*(J(\mathbb{S}^{2n})) \cong \Lambda_{\mathbb{Z}}[\gamma_n] \otimes \Gamma_{\mathbb{Z}}[\gamma_{2n}]$, the tensor product of an exterior algebra on one generator γ_n with a divided polynomial algebra generated by $g_k := \gamma_{2n}^k/k!$.

Hint: See A. Hatcher, Algebraic Topology, pp. 224/225.



The mathematician as young man.