Exercises for Algebraic Topology I

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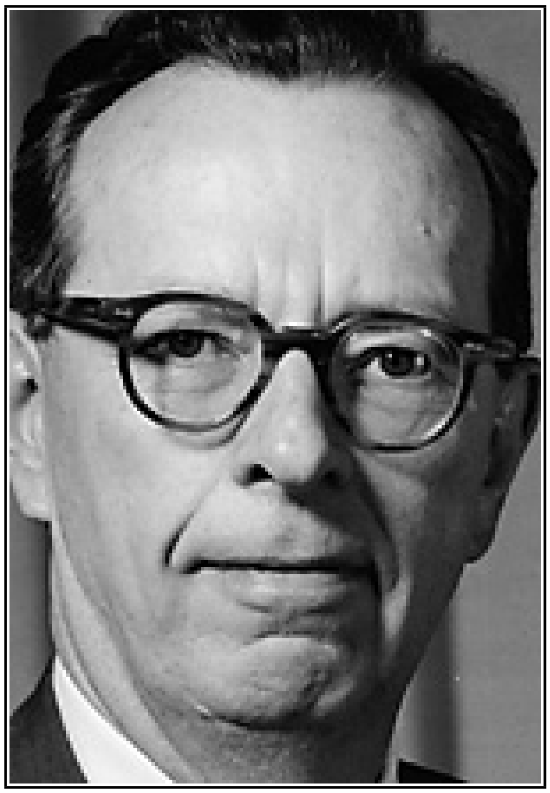
Winter Term 2017/18

Blatt 12 due by: 24.01.2018



George W. Whitehead (1918 - 2004).

Exercise 12.1 (Classifyig space for the symmetric group $\mathfrak{S}_n$)

Find a weakly contractibly space E_n, on which the symmetric group $\mathfrak{S}_n$ acts freely (and properly discontinuously). (Hint: Let $\tilde{C}^n(\mathbb{R}^m)$ be the ordered configuration space of n different points in $\mathbb{R}^m$. Study the higher homotopy groups via the Fadell-Neuwirth fibre bundles

$$\Phi_k^n(\mathbb{R}^m)\colon \tilde{C}^n(\mathbb{R}^m - \{P_1, \ldots, P_k\}) \longrightarrow \mathbb{R}^m - \{P_1, \ldots, P_k\}$$

forgetting the last point of the configuration, which has fibre $\tilde{C}^{n-1}(\mathbb{R}^m - \{P_1, \ldots, P_k, P_{k+1}\})$. Take the limit as m goes to infinity.)

Exercise 12.2 (Compressible and null-homotopic maps)

We call a map $f\colon (X,X') \to (Y,Y')$ of pairs *compressible*, if f is relative to X' homotopic to a map f_1 such that $f_1(X) \subset Y'$.
Show the following two statements:

(a) If f is compressible and X is contractible, then f is null-homotopic.

(b) If f is nullhomotopic and $X' \subset X$ is a cofibration, then f is compressible.

Exercise 12.3 (CW approximation of pairs.)

Recall the CW approximation of a space, i.e., the weak homotopy equivalence $\gamma_X\colon G(X) \to X$ and how it was contracted. Show: If $A \subset X$ we have $G(A) \subset G(X)$ and thus a map of pairs $\gamma_{(X,A)}\colon G(X), G(A)) \to (X,A)$. And show, that $\gamma_{(X,A)}$ induces an isomorphism in homology.

Exercise 12.4 (Some equivalences)

Show two of the following statements:

- $\mathbb{S}^2$ and $\mathbb{S}^3 \times \mathbb{C}P^\infty$ have isomorphic homotipy grops, but nonisomorphic homology (and cohomology groups).
- $\mathbb{S}^m \times \mathbb{R}P^n$ and $\mathbb{S}^n \times \mathbb{R}P^m$ with ($m \neq n$ and $m,n \neq 1$ have isomorphic homotopy groups, but nonisomorphic homology (and cohomology groups).
- $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ have isomorphic homology groups, but nonisomorphic homotopy groups.
- The Hopf map $\eta\colon \mathbb{S}^3 \to \mathbb{S}^2$ induces the trivial homomorphism in reduced homology, but a nontrivial homomorphism in homotopy groups.
- The quotient map $\mathbb{S}^1 \times \mathbb{S}^1 \to (\mathbb{S}^1 \times \mathbb{S}^1)/(\mathbb{S}^1 \vee \mathbb{S}^1) = \mathbb{S}^2$ induces the trivial homomorphism in homotopy groups, but a nontrivial homomorphism in reduced homology groups.

Exercise 12.5* (Another approximation up to homology)

Denote by $\mathfrak{B}_n(X)$ the set of all singular n-simplices $a\colon \Delta^n \to X$ of a space X. Take the disjoint union modulo certain identifications

$$\mathfrak{S}(X) := (\bigsqcup_{n=0}^{\infty} \bigsqcup_{a \in \mathfrak{B}_n(X)} \Delta^n_a)/\sim$$

where the the i-th face of Δ^n_a is linearly identified with $\Delta^{n-1}_{d_i(a)}$, the simplex belonging to $d_i(a) = a \circ d_i$. There is a map $\sigma_X\colon \mathfrak{S}(X) \longrightarrow X$ which is a on Δ^n_a. Show that it is a homology equivalence.