Exercises for Algebraic Topology I

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Winter Term 2017/18

Blatt 11 due by: 17.01.2018

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n \ k	1	2	3	4	5	6	7
2	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$
3	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$
4		$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_{12}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{24}$	$\mathbb{Z}_{15}$
5			$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6				0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$
7					0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$
8						$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_{120}$
9							$\mathbb{Z}_{240}$

D.B. Fuks, V.A. Rochlin: *Beginner's Course in Topology*, p. 437.
The table shows the homotopy groups of spheres $\pi_{n+k}(\mathbb{S}^n)$ for $n = 2, \ldots, 9$ and $k = 1, \ldots, 7$.

Exercise 11.1 (Homotpy groups of projective spaces.)

- Compute all homotopy groups $\pi_q(\mathbb{R}P^\infty)$.
- Compute all homotopy groups $\pi_q(\mathbb{C}P^\infty)$.

Exercise 11.2 (Serre classes of abelian groups)

A *Serre class* $\mathcal{S}$ of abelian groups contains (1) the trivial group, (2) must be closed under subgroups and quotient groups, and (3) contains with A and C the middle group B of any extension $0 \to A \to B \to C \to C \to 0$.

Examples are: a) just the trivial group, b) all finite abelian groups, c) all finitely generated abelian groups, d) all abelian torsion groups, e) all abelian torsion groups with only p-torion.

Show for any Serre class $\mathcal{S}$:

(i) If $C_\bullet$ is a chain complex of abelian groups in $\mathcal{S}$, then all homology groups $H_q(C_\bullet)$ are in $\mathcal{S}$.

(ii) If $F \to E \to B$ is a fibration and both $\pi_q(F)$ and $\pi_q(B)$ are in $\mathcal{S}$ for all q, then all $\pi_q(E)$ is in $\mathcal{S}$ for all q.

(iii) If $A \subset X$ is a subspace and both $H_q(A)$ and $H_q(X, A)$ are in $\mathcal{S}$ for all $q \geq 1$, then $H_q(X)$ is in $\mathcal{S}$ for all $q \geq 1$.

(iv)* If $E^r_{p,q}$ is a first quadrant spectral sequence converging to H_* and if, for some r, all $E^r_{p,q}$ are in $\mathcal{S}$, then H_q is in $\mathcal{S}$ for all q.

Exercise 11.3 (Homotopy groups of a bouquet.)
For $n \geq 2$ show that there is a split short exact sequence of homotopy groups

$$0 \longrightarrow \pi_{n+1}(X \times Y, X \vee Y) \longrightarrow \pi_n(X \vee Y) \longrightarrow \pi_n(X) \oplus \pi_n(Y) \longrightarrow 0..$$

If $X = \mathbb{S}^k$ and $Y = \mathbb{S}^l$, we are tempted to use $\mathbb{S}^k \wedge \mathbb{S}^l \cong \mathbb{S}^{k+l}$ to conclude $\pi_n(\mathbb{S}^k \vee \mathbb{S}^l) = \pi(\mathbb{S}^k) \oplus \pi_n(\mathbb{S}^l) \oplus \pi_n(\mathbb{S}^{k+l})$; is this correct ?

Exercise 11.4 (No quotient isomorphism.)
Here is an example showing $\pi_n(X, A, x_0) \ncong \pi_n(X/A, \bar{x}_0)$ in general.

(a) Compute the relative homotopy group $\pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}, x_0)$ for $n \geq 2$.

(b) Compute the absolute homotopy group $\pi_n(\mathbb{R}P^n/\mathbb{R}P^{n-1}, \bar{x}_0)$ for $n \geq 1$.

Exercise 11.5 (The homotopy group $\pi_7(\mathbb{S}^4)$.)
Prove that $\pi_7(\mathbb{S}^4)$ contains elements of infinite order. (In fact, it is the group $\mathbb{Z} \oplus \mathbb{Z}/12$; see the table above.)

Groups	Generators
Stab(1) = $\pi_4(S^3)$ [$\cong \mathbb{Z}_2$]	su(pr$_*$(sph$_3$))
Stab(2) = $\pi_6(S^4)$ [$\cong \mathbb{Z}_2$]	su^2(pr$_*$(sph$_3$)) $\circ$ su^3(pr$_*$(sph$_3$))
Stab(3) = $\pi_8(S^5)$ [$\cong \mathbb{Z}_{24}$]	su(pr$_*$(sph$_7$))
Stab(4) = $\pi_{10}(S^6)$ [$\cong 0$]	-
Stab(5) = $\pi_{12}(S^7)$ [$\cong 0$]	-
Stab(6) = $\pi_{14}(S^8)$ [$\cong \mathbb{Z}_2$]	su^4(pr$_*$(sph$_7$)) $\circ$ su^7(pr$_*$(sph$_7$))
Stab(7) = $\pi_{16}(S^9)$ [$\cong \mathbb{Z}_{240}$]	su(pr$_*$(sph$_{15}$))

Table 2

k	Stab(k)	k	Stab(k)
8	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	12	0
9	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	13	$\mathbb{Z}_3$
10	$\mathbb{Z}_2$	14	$\mathbb{Z}_6 \oplus \mathbb{Z}_2$
11	$\mathbb{Z}_{504}$	15	$\mathbb{Z}_{480} \oplus \mathbb{Z}_2$

Table 3

Ibidem. — These tables show the stable homotopy groups of spheres $\mathrm{Stab}(k) = \lim_n \pi_{n+k}(\mathbb{S}^n)$ for $k = 1, \ldots, 15$, together with generators for the cases $k = 1, \ldots, 7$.
