Exercises for Algebraic Topology I

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Blatt 10 due by: 20.12.2017

Exercise 10.1 (Relative homotopy groups and homotopy fibres)

Let $\iota\colon A \to X$ be an inclusion of a subspace, $x_0 \in A \subset X$ is the basepoint, and $\mathrm{hFib}(\iota)$ is the homotopy fiber with basepoint $w_0 = (x_0, k)$, where $k = k_{x_0}$, the constant path at x_0. Compare the three homotopy sets

1. $\pi_{n+1}(X; A, x_0) := [(I^{n+1}, \partial I^{n+1}, I^n \times \{0\} \cup \partial I^n \times I)\,;\,(X, A, x_0)]$,
2. $\pi_n(\mathrm{hFib}(\iota), w_0) := [(I^n, \partial I^n)\,;\,(\mathrm{hFib}(\iota), w_0)]$,
3. $[(\mathbb{D}^{n+1}, \mathbb{S}^n, \infty)\,;\,(X, A, x_0)]$.

Define functions

$$D\colon [(I^{n+1}, \partial I^{n+1})\,;\,(X, x_0)] \longrightarrow [(I^{n+1}, \partial I^{n+1}, I^n \times \{0\} \cup \partial I^n \times I)\,;\,(X, A, x_0)]$$

and

$$i\colon [(I^{n+1}, \partial I^{n+1}, I^n \times \{0\} \cup \partial I^n \times I)\,;\,(X, A, x_0)] \longrightarrow [(I^n, \partial I^n)\,;\,(A, x_0)]$$

such that $\mathrm{im}(D) = \ker(i)$.

Exercise 10.2 (Homotopy groups and coverings)

Recall tha following lifting criterion for a based connected covering $\xi\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ of a connected space: a map $f\colon (Y, y_0) \to (X, x_0)$ has lift $\tilde{f}\colon (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ if and only if $\mathrm{im}(f_*\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)) \leq \mathrm{im}(\xi_*\colon \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$. Recall further, that $\pi_1(\mathbb{S}^k, \infty) = 0$ for all $k > 1$; this was an application of. the simplicial approximation theorem for maps.

Prove the following theorem:

If $\pi_k(\tilde{X}) = 0$ for a connected covering space $\tilde{X} \to X$ and some $k \geq 2$, then $\pi_k(X) = 0$.

Check the following examples:

- $X = \mathbb{S}^1$, any $k \geq 2$.
- $X = \mathbb{R}P^n$, for $2 \leq k \leq n$.
- $X = F_g$ or N_g, any orientable or non-orientable surface, and $k \geq 2$.
- $X = \mathbb{C}P^n$, for $2 \leq k \leq 2n$.
- $X = L^n(p, q)$, the $(2n+1)$-dimensional lens spaces, and $2 \leq k \leq 2n$.

(We will soon learn a much more general statement: If $\tilde{X}$ is a connected covering of X, then $\pi_k(\tilde{X}, \tilde{x}_0) \cong \pi_k(X, x_0)$ for all $k \geq 2$.)

Exercise 10.3 (The pull-back of a fibration .....)

...... is a fibration.

Heinz Hopf (rechts) und Hellmuth Kneser

Exercise 10.4 (Some relative homotopy groups)

- Compute $\pi_q(\mathbb{D}^n, \mathbb{S}^{n-1}, \infty)$.
- Compute $\pi_q(V^2, \partial V^2, x_0)$ for the filled torus $V^2 = \mathbb{S}^1 \times \mathbb{D}^2$.
- Compute $\pi_q(M, \partial M, x_0)$ for a contractible manifold M.

Exercise 10.5 (Action of $\pi_1(Y, y_0)$ on based homotopy classes $[X, x_0\,;\, Y, y_0]$)
Assume the inclusion $j\colon \{x_0\} \subset X$ is a cofibration; one calls X *well-based* or the base-poit x_0 *non-degenerate* in this situation. Let y_0 be the basepoint of Y. We want to define an action of $\pi := \pi_1(Y, y_0)$ on the set $[X, Y]_0 := [(X, x_0)\,;\,(Y, y_0)]$ of based homotopy classes. More generally, let $w\colon I \to Y$ be any path from $w(0) = y_0$ to $w(1) = y_1$. Take a map $h\colon X \to Y$ with $h(x_0) = y_0$ and consider the partial homotopy $H\colon X \times \{0\} \cup \{x_0\} \times I \to Y$ defined by $H(x, 0) := h(x)$ and $H(x_0, t) := w(t)$. By the defining property of a cofibration we can extend H to a map full homotopy $\bar{H}\colon X \times I \to Y$. The map $x \mapsto \bar{H}(x, 1)$ sends x_0 to y_1. We denote it by $w_\#(h)$, — although it depends not only on w and h, but on a choice for the homotopy $\bar{H}$; to be precise, $w_\#(h)$ can mean any map at the top of

any homotopy $\bar{H}$ which extends H. We claim that the function

$$T_w\colon [(X,x_0)\,;\,(Y,y_0)] \longrightarrow [(X,x_0)\,;\,(Y,y_1)], \quad [h] \mapsto [w_\#(h)]$$

is well-defined and a bijection of pointed sets. So show the following properties:

1. If $f \simeq g \operatorname{rel} x_0$, then $w_\#(f) \simeq w_\#(g) \operatorname{rel} x_0$.
2. If $w \simeq v \operatorname{rel} \partial I$, then $w_\#(f) \simeq v_\#(f) \operatorname{rel} x_0$.
3. $T_v \circ T_w = T_{w\star v}$ for w a path from y_0 to y_1 and v a path from y_1 to y_2 and $w \star v$ their product.
4. $T_{k_{y_0}} = \mathrm{id}$ for the constant path k_{y_0} at y_0.
5. $T_{\bar{w}} = (T_w)^{-1}$ for the reverse path $\bar{w}(t) = w(1-t)$.
6. $T_w([c_{y_0}]) = [c_{y_1}]$ for the constant maps to y_0 resp. to y_1.

So for $y_0 = y_1$ we have a group action

$$\pi := \pi_1(Y,y_0) \times [X,Y]_0 \longrightarrow [X,Y]_0.$$

(For $X = \mathbb{S}^n$, $x_0 = \infty$ this is the action we defined on $\pi_n(Y,y_0)$.)
Show that the quotient set of this action is $[X,Y]$.

Exercise 10.6* (Action of $\pi_1(Y,y_0)$ on based homotopy classes $[X,x_0\,;\,Y,y_0]$, continued)
We continue to assume that X is well-based. Consider a two maps $f_0, f_1\colon X \to Y$ and a free homotopy $F\colon X\times I \to Y$ with $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in X$. Evaluating at the basepoint x_0 we obtain a track curve $w(t) := F(x_0,t)$ in Y from $y_0 := f_0(x_0)$ to $y_1 := f_1(x_0)$. We say f_0 and f_1 are *freely homotopic with track curve w*, denoted $f_0 \simeq_w f_1$. Show the following:

(a) Given f_0 and some path w with $w(0) = f_0(x_0)$, there is some f_1 with $f_0 \simeq_w f_1$.

(b) If $f_0 \simeq_w f_1$ and $f_0 \simeq_v f_2$ and $w \simeq v \operatorname{rel} \partial I$, then $f_1 \simeq_c f_2$, where c is the constant path.

(c) If $f_0 \simeq_w f_1$ and $f_1 \simeq_v f_2$, then $f_0 \simeq_{w\star v} f_2$.

This defines an equivalence relation on $[X,Y]_0$.

(d) The set of equivalence classes is $[X,Y]$.

And conclude:

(e) If X and Y are path-connected, then a based map $f\colon (X,x_0) \to (Y,y_0)$ of is freely null-homotopic if and only if it is based homotopic.

(f) The relaxation map $[X,Y]_0 \to [X,Y]$ is a bijection, if Y is connected and simply-connected.