Exercises for Algebraic Topology I

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Blatt 9 due by: 13.12.2017

Exercise 9.1 (Homotopy equivalences)

Show the equivalence of the following three statements:

(i) $f\colon X \to Y$ is a homotopy equivalence.

(ii) $f_*\colon [A, X] \to [A, Y]$ is a bijection, for all spaces A.

(iii) $f^*\colon [Y, E] \to [X, E]$ is a bijection, for all spaces E.

Of course, there is a based and an unbased version.

Exercise 9.2 (Neighbourhood Deformation Retraction)

Let $\iota\colon A \subset X$ be an inclusion. One calls $A \subset X$ a *neighbourhood deformation retract (NDR)* if there is a continuous function $u\colon X \to [0, 1]$ and a homotopy $\rho\colon X \times I \to X$ such that

1. $A = u^{-1}(0)$,
2. $\rho(x, 0) = x$ for all $x \in X$,
3. $\rho(a, t) = a$ for all $a \in A$ and all $t \in I$,
4. $\rho(x, t) \in A$ for all $t > u(x)$.

We know from the lecture, that an NDR is a cofibration. Now we want to prove the opposite direction: If ι is a cofibration, then it is a NDR.

(Hint: Use that there is a retraction $R\colon X \times I \to (A \times I) \cup (X \times \{0\})$ of the mapping cylinder and consider the functions $u(x) := \max\{t - p_2(R(x, t)) \mid t \in I\}$ and $\rho(x, t) := p_1(R(x, t))$, where p_1 resp. p_2 denote the projection of $X \times I$ to X resp. to I.)

Exercise 9.3 (Explicit homotopy equivalence between mapping cone and quotient)

Assume $A \subset X$ is a cofibration. By Exerc. 9.2 it is a NDR and we use the notation of this exercise. Consider the function $\tilde{s}\colon X \to (A \times I) \cup (X \times \{0\})$ sending $x \mapsto (\rho(x, 1), 1 - u(t))$. Show that it induces a map

$$s\colon X/A \longrightarrow (A \times I) \cup (X \times \{0\})/A \times \{1\} \cong C(\iota),$$

which is a homotopy inverse to the quotient map $\lambda\colon C(\iota) \longrightarrow C(\iota)/C(A) \cong X/A$.

Exercise 9.4 (Puppe sequence)

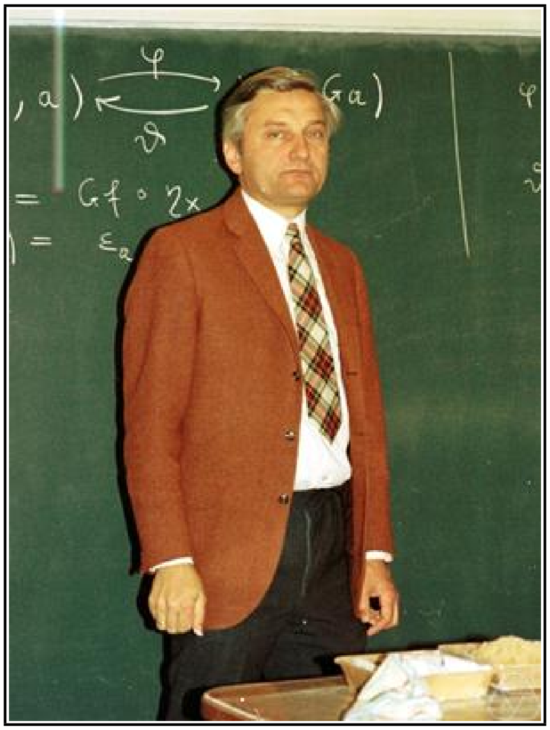
Consider the Puppe sequence or cofiber sequence

$$\mathbb{S}^1 \xrightarrow{f} \mathbb{S}^1 \xrightarrow{j} \mathbb{R}P^2 \xrightarrow{q} \mathbb{S}^2 \xrightarrow{\Sigma f} \mathbb{S}^2 \xrightarrow{\Sigma j} \Sigma\mathbb{R}P^2 \xrightarrow{\Sigma q} \mathbb{S}^3 \longrightarrow \dots$$

of a map f of degree 2. Consider the five terms of the exact sequence of pointed sets of based homotopy classes $[\ \ , E]$ of maps to a space E:

$$[\mathbb{S}^1, E] \xleftarrow{f^*} [\mathbb{S}^1, E] \xleftarrow{j^*} [\mathbb{R}P^2, E] \xleftarrow{q^*} [\mathbb{S}^2, E] \xleftarrow{(\Sigma f)^*} [\mathbb{S}^2, E]$$

By setting $E = \mathbb{S}^1$, $E = \mathbb{S}^2$, $E = \mathbb{S}^k$ for $k \geq 3$ and $E = \mathbb{R}P^2$, conclude that



Dieter Puppe (1930 - 2005)

1. All maps from $\mathbb{R}P^2$ to $\mathbb{S}^1$ are inessential.
2. All essential maps from $\mathbb{R}P^2$ to $\mathbb{S}^2$ are homotopic to q.
3. All maps from $\mathbb{R}P^2$ to $\mathbb{S}^k$ with $k \geq 3$ are inessential.

Can we prove similar statements for $\mathbb{R}P^n$?