A CONVENIENT CATEGORY OF TOPOLOGICAL SPACES

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Dedicated to R. L. Wilder, who taught my first course in analysis situs, suggested my first research problem, and nursed my initial efforts to fruition.

1. INTRODUCTION

For many years, algebraic topologists have been laboring under the handicap of not knowing in which category of spaces they should work. Our need is to be able to make a variety of constructions and to know that the results have good properties without the tedious spelling out at each step of lengthy hypotheses such as countably paracompact, normal, completely regular, first axiom of countability, metrizable, and so forth. It may be good research technique and an enjoyable exercise to analyse the precise circumstances for which an argument works; but if a developing theory is to be handy for research workers and attractive to students, then simplicity of the fundamentals must be the goal.

The demands which a *convenient* category should satisfy are first that it be large enough to contain all of the particular spaces arising in practice. Second, it must be closed under standard operations; these are the formation of subspaces, product spaces $X \times Y$, function spaces Y^X, decomposition spaces, unions of expanding sequences of spaces, and compositions of these operations. Third, the category should be small enough so that certain reasonable propositions about the standard operations are true. These state that the order of performing two operations can be reversed. We adopt the following as test propositions.

(1) $$(Y \times Z)^X = Y^X \times Z^X .$$

(2) $$Z^{Y \times X} = (Z^Y)^X .$$

(3) A product of decomposition spaces is a decomposition space of the product.

(4) A product of unions is a union of products.

(5) A decomposition space of a union is a union of decomposition spaces.

It is well known that (1), (2), and (3) are valid for compact metric spaces, but the category of these is not closed, under several standard operations. It is also known that these propositions do not hold in the category of all Hausdorff spaces. In fact, arguments have been given which imply that there is no convenient category in our sense [13, Appendix]. The arguments are based on a blind adherence to the customary definitions of the standard operations. These definitions are suitable for the category of Hausdorff spaces, but they need not be for a subcategory. The categorical viewpoint enables us to defrost these definitions and bend them a bit.