# Exercises for Algebraic Topology I

Prof. Dr. C.-F. Bödigheimer

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**Blatt 8** due by: 6.12.2017

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Norman Steenrod (1910 - 1971)

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**Exercise 8.1** (Homotopy-fiber.)

(a) Show that a map $f\colon X \to Y$ is null-homotopic if and only if it factors through some path space of $Y$, i.e., if and only if there exists a map $\tilde{f}\colon X \to P(Y, y_0)$ such that $\mathrm{ev}_1 \circ \tilde{f} = f$.
Here $P(Y, y_0)$ is the space of all paths $w\colon [0,1] \to Y$ starting at $y_0$; the evaluation map $\mathrm{ev}_1\colon P(Y, y_0) \to Y$ is $\mathrm{ev}_1(w) = w(1)$.

(b) Let $f\colon X \to Y$ be a map into a path-connected space $Y$ and let $g\colon Z \to X$. Show that $f \circ g$ is null-homotopic if and only if $g$ factors through the canonical map $p\colon \mathrm{hFib}(f) \to X$, i.e., there is a map $\tilde{g}\colon Z \to \mathrm{hFib}(f)$ such that $p \circ \tilde{g} = g$.
Note that the homotopy-fiber $\mathrm{hFib}(f)$ or better say the map $p\colon \mathrm{hFib}(f) \to X$ is the pull-back of the evaluation

$\mathrm{ev}_1 \colon P(Y, y_0) \to Y$ for some $y_0 \in Y$:

$$\begin{tikzcd}
 & \mathrm{hFib}(f) \arrow[r] \arrow[d, "p"'] & P(Y, y_0) \arrow[d, "\mathrm{ev}_1"'] \\
Z \arrow[ur, dashed, "\tilde{g}"] \arrow[r, "g"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

**Exercise 8.2** (Push-forward of a cofibration)
Let $\iota \colon A \to X$ be a cofibration and $g \colon A \to B$ any map, then $\iota' \colon B \to Y = B \cup_g X$ is a cofibration.

**Exercise 8.3** (Formal properties of Push-outs and pull-back squares)
Let $\mathcal{C}$ be any category. (Note: Push-outs and pull-backs have a universal property with uniqueness. )
(a) Consider the square
$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\
C \arrow[r] & C
\end{tikzcd}$$
and prove two of the following statements:

(a.1) If the square is a push-out and $f$ is an isomorphism, then $g$ is an isomorphism.

(a.2) If $f$ and $g$ are isomorphismsm then the square is a push-out.

(a.3) State and prove the dual statements of (a.1) and (a.2).

(b) Consider in the commutative diagram
$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "h_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"] & A_3 \arrow[d, "h_3"] \\
B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3
\end{tikzcd}$$
and denote the left square by (I), the right square by (II) and the outer square by (III); prove one of the following statements:

(b.1) If (I) and (II) are push-outs, then (III) is a push-out.

(b.2) If (I) and (III) are push-outs, then (II) is a push-out.

(c) Consider the same commutative diagram and prove one of the following statements:

(c.1) If (I) and (II) are pull-backs, then (III) is a pull-back.

(c.2) If (II) and (III) are pull-backs, then (I) is a pull-back.

**Exercise 8.4** (Compactly-generated hausdorff spaces)

(1) Show: A first-countable hausdorff space is compactly-generated.

(2) Show: If $f \colon X \to Y$ is a quotient map of hausdorff spaces and $X$ is compactly-enerated, then $Y$ is compactly-generated (but in general not hausdorff).

**Exercise 8.5*** (Compactly generated weak hausdorff spaces)
A topological space $X$ is called *weakly-hausdorff* if $f(K) \subset X$ is closed in $X$ for any continuous map $f \colon K \to X$ from a compact space $K$ to $X$. This is equivalent to the diagonal $\Delta(X)$ being closed in $X \times_k X = k(X \times X)$, the compactly-generated refinement of the product topology of $X \times X$.

## A CONVENIENT CATEGORY OF TOPOLOGICAL SPACES

N. E. Steenrod

Dedicated to R. L. Wilder, who taught my first course in analysis situs, suggested my first research problem, and nursed my initial efforts to fruition.

### 1. INTRODUCTION

For many years, algebraic topologists have been laboring under the handicap of not knowing in which category of spaces they should work. Our need is to be able to make a variety of constructions and to know that the results have good properties without the tedious spelling out at each step of lengthy hypotheses such as countably paracompact, normal, completely regular, first axiom of countability, metrizable, and so forth. It may be good research technique and an enjoyable exercise to analyse the precise circumstances for which an argument works; but if a developing theory is to be handy for research workers and attractive to students, then simplicity of the fundamentals must be the goal.

The demands which a *convenient* category should satisfy are first that it be large enough to contain all of the particular spaces arising in practice. Second, it must be closed under standard operations; these are the formation of subspaces, product spaces $X \times Y$, function spaces $Y^X$, decomposition spaces, unions of expanding sequences of spaces, and compositions of these operations. Third, the category should be small enough so that certain reasonable propositions about the standard operations are true. These state that the order of performing two operations can be reversed. We adopt the following as test propositions.

(1) $$(Y \times Z)^X = Y^X \times Z^X .$$

(2) $$Z^{Y \times X} = (Z^Y)^X .$$

(3) A product of decomposition spaces is a decomposition space of the product.

(4) A product of unions is a union of products.

(5) A decomposition space of a union is a union of decomposition spaces.

It is well known that (1), (2), and (3) are valid for compact metric spaces, but the category of these is not closed, under several standard operations. It is also known that these propositions do not hold in the category of all Hausdorff spaces. In fact, arguments have been given which imply that there is no convenient category in our sense [13, Appendix]. The arguments are based on a blind adherence to the customary definitions of the standard operations. These definitions are suitable for the category of Hausdorff spaces, but they need not be for a subcategory. The categorical viewpoint enables us to defrost these definitions and bend them a bit.

N. Steenrod: *A convenient category of topological spaces*, Michigan Math. Journal, vol. 14 (1967), p. 133.

Let $\mathcal{CG}$ denote the category of compactly-generated spaces weakly-hausdorff spaces and continuous maps, and let $\mathcal{WH}$ denote the category of weakly-hausdorff spaces and continuous maps. We consider the forgetful functor $j\colon \mathcal{CG} \to \mathcal{WH}$ and the refinement functor $k\colon \mathcal{WH} \to \mathcal{CG}$.
Show that there is a bijection of sets

$$\mathrm{map}(X, k(Y)) \cong \mathrm{map}((j(X), Y)$$

for $X \in \mathcal{CG}$ and $Y \in \mathcal{WH}$. Thus the two functors are adjoint.