Exercises for Algebraic Topology I

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Blatt 7 (now complete) due by: 29.11.2017

A round man will not fit through a square hole; you must give him time to change. — Mark Twain

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Exercise 7.1 (Not null-homotopic maps.)

1. If $f\colon X \to Y$ is null-homotopic, then the mapping cone $C(f) \simeq \Sigma X \vee Y$ is a wedge.
2. If $Z \simeq Z_1 \vee Z_2$, then $\tilde{H}^*(Z;\mathbb{K}) \cong \tilde{H}^*(Z_1;\mathbb{K}) \oplus \tilde{H}^*(Z_2;\mathbb{K})$ as rings without unit, for any commutative ring $\mathbb{K}$.
3. Example: The Hopf map $\eta\colon \mathbb{S}^3 \to \mathbb{S}^2 = \mathbb{C}P^1$ is not null-homotopic.
4. Example: A degree 2 map $\epsilon\colon \mathbb{S}^1 \to \mathbb{S}^1 = \mathbb{R}P^1$ is not null-homotopic. (*Yes, I know — it follows from the degree not being zero; but this proof here fits so beautifully as an example.*)
5. Examples: The attaching map $f\colon \mathbb{S}^{2n-1} \to \mathbb{C}P^{n-1}$ of the top 2n-cell in $\mathbb{C}P^n = \mathbb{C}P^{n-1} \cup_f e^{2n}$ is not null-homotopic.
6. Examples: The attaching map $f\colon \mathbb{S}^{n-1} \to \mathbb{R}P^{n-1}$ of the top n-cell in $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup_f e^n$ is not null-homotopic.

Exercise 7.2 (Homotopy invariance of homotopy push-outs.)
For a diagram of three spaces and maps as

$$\begin{tikzcd} X_2 & \\ X_0 \arrow[u, "\iota_2"] \arrow[r, "\iota_1"'] & X_1 \end{tikzcd}$$

we have defined the *homotopy push-out* (or *double mapping cylinder*) $Z := \mathrm{hocolim}(\iota_1, \iota_2)$ together with maps j_1 and j_2 in a homotopy-commutative square

$$\begin{tikzcd} X_2 \arrow[r, "j_1"] & Z \\ X_0 \arrow[u, "\iota_2"] \arrow[r, "\iota_1"'] & X_1 \arrow[u, "j_2"'] \end{tikzcd}$$

satisfying a universal property. Consider the strictly commutative diagram of spaces and maps:

$$\begin{tikzcd} X_1 \arrow[d, "f_1"'] & X_0 \arrow[l, "\iota_1"'] \arrow[r, "\iota_2"] \arrow[d, "f_0"'] & X_2 \arrow[d, "f_2"'] \\ Y_1 & Y_0 \arrow[l, "\kappa_1"] \arrow[r, "\kappa_2"'] & Y_2 \end{tikzcd}$$

and show that there is a continous function

$$\zeta(f_1, f_0, f_2)\colon \operatorname{hocolim}(\iota_1, \iota_2) \to \operatorname{hocolim}(\kappa_1, \kappa_2)$$

with the following properties:

(1) $\zeta(\mathrm{id}_{X_1}, \mathrm{id}_{X_0}, \mathrm{id}_{X_2}) = \mathrm{id}_Z$.

(2) $\zeta(g_1 \circ f_1, g_0 \circ f_0, g_2 \circ f_2) = \zeta(g_1, g_0, g_2) \circ \zeta(f_1, f_0, f_2)$.

(3) If f_1^s, f_0^s and f_1^s are three homotopies ($0 \le s \le 1$) with the obvious compatabilties, then $\zeta_s := \zeta(f_1^s, f_0^s, f_2^s)$ is a homotopy.

(4) If f_1, f_2 and f_0 are homotopy-equivalences, then $\zeta(f_1, f_0, f_2)$ is a homotopy-equivalence. (So the homotopy colimit is a homotopy-invariant construction.)

(5)* Show in contrast, that the topological pushout $C := X_1 \cup_{X_0} X_2$ is not a homotopy invariant construction: look at the diagram with $X_1 = \mathbb{D}^2$, $X_0 = \mathbb{S}^1$ and $X_2 = *$ in the top row and $Y_1 = \mathbb{R}^2$, $Y_0 = \mathbb{R}^2 - 0$ and $Y_2 = *$ in the bottom row and all maps being the obvious inclusions or constant maps.

Exercise 7.3 (Composition and homotopy colimits)
Consider the diagram

$$\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[d, "\alpha"'] & X_1 \arrow[r, "g"] \arrow[d, "\beta"] & X_2 \arrow[d, "\gamma"] \\ Y_0 \arrow[r, "\tilde{f}"'] & Y_1 \arrow[r, "\tilde{g}"'] & Y_2 \end{tikzcd}$$

where the left square and the right square are homotopy push-outs, so $Y_1 = \operatorname{hocolim}(f, \alpha)$ and $Y_2 = \operatorname{hocolim}(g, \beta)$. Show that the outer square is a homotopy pushout and thus $\operatorname{hocolim}(g, \beta) \simeq \operatorname{hocolim}(g \circ f, \alpha)$.

Exercise 7.4 (Free and based homotopies)
Show the relaxation function from based homotopy classes to free homotopy classes

$$R\colon [(X, x_0), (Y, y_0)] \longrightarrow [X, Y]$$

to be surjective for path-connected spaces X and Y. Find an example, where it is not injective.

Here the email exchange with a student on 22.11.2017:

Lieber Herr Bödigheimer,
wollen wir bei Aufgabe 4 des aktuellen Blattes nicht noch annehmen, daß auf X die Inklusion des Basispunktes eine Kofaserung ist?
Beste Grüße, FK

Lieber Herr K.,
ja, wir wollen und wir müssen. - Nun wissen wir leider offiziell noch nicht, was eine Kofaserung ist. (Das sage ich am Montag.)
Deshalb müsste man voraussetzen: *Assume that there is a neighbourhood U of $x_0 \in X$ and a homotopy $r_t\colon X \to X$ and so on.* Man könnte vielleicht einfach sagen: *Assume that x_0 has a neighbourhood U in X, which is homeomorphic to a disk $\mathbb{D}^m$ for some m.* Oder wir sagen: *Replace X by the double mapping cylinder of $x_0 \leftarrow x_0 \to X$* . (d.h. hefte an x_0 einen Stachel an), was homotopie-äquivalent zu X ist. --- Sie müssen zugeben, dass die alte Aufgabe mit der fehlenden Voraussetzung viel schöner ist und besser klingt. Nun gut, ich korrigiere das morgen.
CFB

Poincaré, leaving the scene.

Exercise 7.5* (Farewell to Poincaré Duality.)
(a) Assume the connected, compact, closed and orientable m-manifold M is triangulated as the geometric realization $M = |\mathcal{X}|$ of a finite polyhedron $\mathcal{X}$; we denote the barycentric subdivision of $\mathcal{X}$ by $\mathcal{X}'$. For any simplex $\sigma \in \mathcal{X}_q$ let $D(\sigma)$ denote its *dual block* by $D(\sigma)$, the closed star of σ with respect to the subdivision $\mathcal{X}'$, which is defined as the union of simplices of in $\mathcal{X}'$ or flags $\phi = (\sigma_0 < \sigma_1 < \ldots < \sigma_m)$ in $\mathcal{X}$ starting with $\sigma_0 = \sigma$.
Each $D(\sigma)$ is homeomorphic to a closed disc of dimension $m - q$. Since we have

$$D((v_0, v_1, \ldots, v_q)) = D(v_0) \cap D(v_1) \cap \ldots \cap D(v_q)$$

for $\sigma = (v_0, v_1, \ldots, v_q)$ spanned by the vertices $v_0, v_1, \ldots, v_q \in \mathcal{X}_0$ of $\mathcal{X}$, the dual blocks form a cellular (not simplicial) structure $\mathcal{S}$ on M.
Let $\mathcal{C}_\bullet(\mathcal{X})$ denote the simplicial chain complex of the simplicial structure $\mathcal{X}$, with boundary operator ∂. And let $\mathcal{B}^\bullet(\mathcal{S})$ be the cochain complex of the cellular structure $\mathcal{S}$, with coboundary operator δ. The orientation of M determins the signs of faces and co-faces in ∂ resp. δ. The function

$$D\colon \mathcal{C}_q(\mathcal{X}) \longrightarrow \mathcal{B}^{m-q}(\mathcal{S}), \quad \sigma \mapsto D(\sigma)^\#,$$

where $D(\sigma)^\#(\tau) = 1$, if $\tau = \sigma$, and $= 0$, if $\tau \neq \sigma$, extends to an isomorphism of free modules. We have the duality property

$$\delta D(a) = D(\partial a)^\#.$$

in other words

$$\begin{tikzcd}
C_{m-q+1} & B^{m-q+1} \\
C_{q-m} & B^{q-m} \arrow[u, "\delta"'] \\
C_q \arrow[ur, "D"] \arrow[d, "\partial"'] & B^q \\
C_{q-1} \arrow[uuur, "D"'] & B^{q-1}
\end{tikzcd}$$

Show that D induces an isomorphsm

$$H_q(M) \cong H_q(\mathcal{C}_\bullet(\mathcal{X})) \longrightarrow H^{m-q}(\mathcal{B}^\bullet(\mathcal{S})) \cong H^{m-q}(M).$$

(b) Now, where is the cap product and the fundamental class ? — Let $C^\bullet(\mathcal{X}')$ be the simplicial cochain complex of the simplicial stucture $\mathcal{X}'$, with coboundary operator δ'. And on the other side we take $\mathcal{C}_\bullet(\mathcal{X}')$ to be the simplicial chain complex of $\mathcal{X}'$, with boundary operator ∂'. The fundamental class $[M]$ is the signed sum over all m-simplices in $\mathcal{X}'$, thus over all flags $\phi = (\sigma_0 < \sigma_1 < \ldots < \sigma_m)$ with all $\sigma_i \in \mathcal{X}_i$ simplices of the old triangulations of dimension i; the signs are determined by the given orientation.
Recall that the simplicial version of the cap produrct is given by the formula

$$\alpha \cap (\sigma_0 < \ldots < \sigma_q) = \alpha(((\sigma_0 < \ldots < \sigma_k)) \cdot (\sigma_k < \ldots < \sigma_q)$$

for a simplicial k-cochain α and a q-simplex $b = (\sigma_0 < \ldots < \sigma_k < \ldots < \sigma_q)$ with $k \leq q$. Applied to the fundamental class $b = [M] = \sum \pm(\sigma_0 < \sigma_1 < \ldots < \sigma_m)$ we obtain

$$\alpha \cap [M] = \sum \pm \alpha((\sigma_0 < \ldots < \sigma_q)) \cdot (\sigma_k < \ldots < \sigma_m).$$

So if $\alpha = (\sigma_0 < \ldots < \sigma_q)^\#$ is the dual form of the simplex $(\sigma_0 < \ldots < \sigma_q)$, then $\alpha \cap [M]$ is the signed sm of all $(m-q)$-simplices $\psi = (\sigma_q < \ldots < \sigma_m)$, since $(\sigma_0 < \ldots \sigma_q \ldots \sigma_m)$ occurs in the fundamental class $[M]$.
The boundary formula for he cap product is

$$\partial'(\alpha \cap b) = \delta'(\alpha) \cap b \ \pm \ \alpha \cap \partial'(b)$$

If we fix $b = [M]$, which is a cycle, we obtain a duality

$$P \colon \mathcal{C}^q(\mathcal{X}') \longrightarrow \mathcal{C}_{m-q}(\mathcal{X}'), \quad P(\alpha) = \alpha \cap [M].$$

Show, that P induces an isomorphism

$$H^q(M) \cong H^q(\mathcal{C}^\bullet(\mathcal{X}')) \longrightarrow H_{m-q}(\mathcal{C}_\bullet(\mathcal{X}')) \cong H_{m-q}(M).$$