Exercises for Algebraic Topology I

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Blatt 6 due by: 22.11.2017

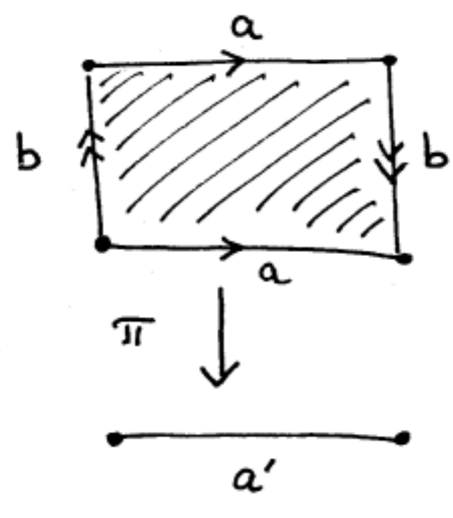
Exercise 6.1 (Cohomology of the Klein bottle.)
The Klein bottle K is an $\mathbb{S}^1$-bundle $\pi\colon K \to \mathbb{S}^1$ over $\mathbb{S}^1$; see figure below.

(1) Compute the integral homology using the cell structure from the figure.

(2) Compute the integral cohomology groups.

(3) Determine the ring structure. (Note that π^* is injective.)

(4)* Analyze the Gysin sequence in mod-2 cohomology.



Klein bottle.

Exercise 6.2 (The unitary group $\mathbf{U}(2)$ fibred in two different ways.)
The unitary group $\mathbf{U}(2)$ is fibred as the second Stiefel manifold $V_2(\mathbb{C}^2) = \mathbf{U}(2)$ via

$$\mathbb{S}^1 \longrightarrow \mathbf{U}(2) \longrightarrow \mathbb{S}^3, \quad A = (v_1, v_2) \mapsto v_2,$$

and then a second time with the determinant as

$$\mathbb{S}^3 \cong \mathbf{SU}(2) \longrightarrow \mathbf{U}(2) \longrightarrow \mathbb{S}^1, \quad A \mapsto \det(A).$$

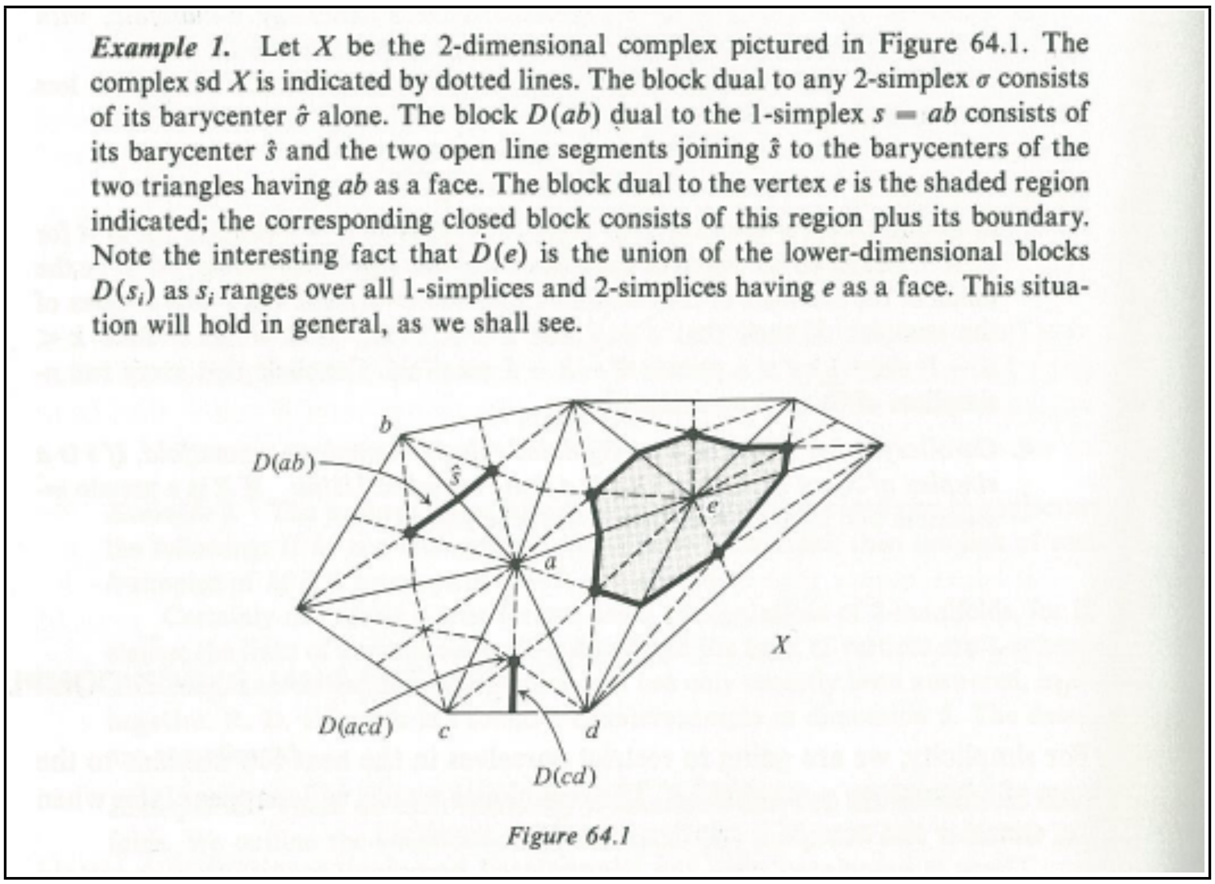
Analyse the Gysin sequence of these sphere bundles: Are the maps group homomorphisms ? Are there sections ? What are the Euler classes ? How can we compare the results ? The cohomology looks as if $\mathbf{U}(2)$ were the product $\mathbb{S}^1 \times \mathbb{S}^3$.

Exercise 6.3 (Self-intersections.)

Let $i\colon N \to M$ be the inclusion of a connected, closed, compact and oriented submanifold of dimension n into a connected, compact and oriented manifold of dimension m. We denote by $[N] \in H_n(N)$ the fundamental class and by $[N]_M := i_*([N]) \in H_n(M)$ its image in the homology of M. A *vector field on N in M* is a section V of $T(M)|N \to N$, the restriction of the tangent bundle of M to N; we say V is *transverse* to N, if $V(z) \notin T_z(N)$ for all $z \in N$.

Prove: If $[N]_M \bullet [N]_M \neq 0$, then there is no non-vanishing vector field on N in M transverse to N.

(*Example*: Although the Klein bottle K (see Exercise 6.1 and the figure) is not orientable, the two homology classes a and b in $H_1(K)$ illustrate the possibilities of self-intersections.)



Example 1. Let X be the 2-dimensional complex pictured in Figure 64.1. The complex sd X is indicated by dotted lines. The block dual to any 2-simplex σ consists of its barycenter $\hat{\sigma}$ alone. The block $D(ab)$ dual to the 1-simplex $s = ab$ consists of its barycenter $\hat{s}$ and the two open line segments joining $\hat{s}$ to the barycenters of the two triangles having ab as a face. The block dual to the vertex e is the shaded region indicated; the corresponding closed block consists of this region plus its boundary. Note the interesting fact that $\dot{D}(e)$ is the union of the lower-dimensional blocks $D(s_i)$ as s_i ranges over all 1-simplices and 2-simplices having e as a face. This situation will hold in general, as we shall see.

Figure 64.1

J.R. Munkres: *Elements of Algebraic Topology*, p. 378.
Illustration of the block decomposition of a triangulated manifolds for the geometric proof of the Poincaré duality.

Exercise 6.4* (Čech cohomology is singular cohomology for an ENR.)

Let X be a ENR in some $\mathbb{R}^n$, so $X \subset W \subset \mathbb{R}^n$ with W open in $\mathbb{R}^n$ and $r\colon W \to X$ a retraction. For all neighbourhoods U of X in W the homomorphisms $j^U_*\colon H^q(U) \to H^q(X)$ induced by the inclusions $j^U\colon X \to U$

Eduard Cech (1893 - 1960), hier auf einer Tagung 1935 in Moskau.

form a coherent system and thus induce a map

$$J \colon \check{H}^q(X) = \operatorname{limdir}_U H^q(U) \longrightarrow H^q(X)$$

from the Čech cohomology to the singular cohomology. Show that J is an isomorphism.
(Hint: See G.E. Bredon: *Geometry and Topology*, p. 539.).

Example: Any compact manifold is an ENR:
(1) Any compact m-manifold M can be embedded into some euclidean space. For a proof one collapses the complements of open charts $U_i \cong \mathbb{D}^m$, for $i = 1, \dots, r$, to a point, obtaining a continous map $\psi_i \colon M \to M/(M - U_i) \cong \mathbb{S}^m \subset \mathbb{R}^{m+1}$; then $\Psi = (\psi_1, \dots, \psi_r) \colon M \to \mathbb{R}^{(m+1)r}$ is an embedding; see A. Dold: *Lectures on Algebraic Topology*, p. 82.
(2) Then one uses a Theorem of Borsuk: Any locally compact and locally contractible subset $X \subset \mathbb{R}^N$ is an ENR. See A. Dold, loc. cit., p. 83; or G.E. Bredon, loc. cit., p. 537, for a proof.