Exercises for Algebraic Topology I

Prof. Dr. C.-F. Bödigheimer

Winter Term 2017/18

Blatt 5 due by: 15.11.2017

Exercise 5.1 (Thom spaces)

For a k-dimensional vector bundle $\xi\colon E \to B$ we denote disk bundle by $D(E)$ and the sphere bundle by $S(E)$. Their quotient $\mathrm{Th}(\xi) := D(E)/S(E)$ we call the *Thom space* of ξ. (Note that $D(E)$ and $S(E)$ do depend on the choice of a metric on the bundle; but different choices lead to homeomorphic Thom spaces.).

Rene Thom (1923 - 2002), french topologist.

Show two of the following statements (1) - (4).

(0) $\mathrm{Th}(\xi) \cong B_+ \wedge \mathbb{S}^k$, if ξ is a trivial bundle.

(1) $\mathbb{R}P^{n+1}$ is the Thom space of the dual of the canonical line bundle over $\mathbb{R}P^n$. Thus $\mathbb{R}P^{n+1} - \{x\} \simeq \mathbb{R}P^n$.

(2) (**Whitney sum**) $\mathrm{Th}(\xi \oplus \zeta) \cong \mathrm{Th}(\xi) \wedge \mathrm{Th}(\zeta)$ for the sum of two vector bundles.

(3) (**Functoriality**) For any injective bundle map $(\tilde{f}, f)\colon \xi \to \zeta$ there is a map $\mathrm{Th}(\tilde{f}, f)\colon \mathrm{Th}(\xi) \to \mathrm{Th}(\zeta)$ such that

(a) $\mathrm{Th}(\mathrm{id}_E, \mathrm{id}_B) = \mathrm{id}_{\mathrm{Th}(\xi)}$ and

(b) $\mathrm{Th}(\tilde{g} \circ \tilde{f}, g \circ f) = \mathrm{Th}(\tilde{g}, g) \circ \mathrm{Th}(\tilde{f}, f)$.

Examples:
Grassmann vector bundles $E_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$ over Grassmann manifolds with $f\colon \mathrm{Gr}_k(\mathbb{R}^n) \to \mathrm{Gr}_{k+l}(\mathbb{R}^{n+l})$ sending the linear subspace $L \subset \mathbb{R}^n$ to $L \subset \mathbb{R}^n \oplus \mathbb{R}^l$, or $g\colon \mathrm{Gr}_k(\mathbb{R}^n) \to \mathrm{Gr}_{k+l}(\mathbb{R}^{n+l})$ sending L to $L \oplus \mathrm{Span}(e_{n+1}, \ldots, e_{n+l}) \subset \mathbb{R}^{n+l}$, both with corresponding map $\tilde{f}\colon E_k(\mathbb{R}^n) \to E_r(\mathbb{R}^{n+l})$ resp. $\tilde{g}\colon E_k(\mathbb{R}^n) \to E_{k+l}(\mathbb{R}^{n+l})$ on the total spaces.

(4) (**Naturality of the Thom class**) $\mathrm{Th}(\tilde{f}, f)^*(\tau(\zeta)) = \tau(f^*\zeta)$ for the Thom class τ of a bundle $\zeta\colon E' \to B'$ and of its pull-back $\xi := f^*\zeta\colon E := f^*E' \to B$ along any map $f\colon B \to B'$, and $\tilde{f}\colon E \to E'$ the induced map.

Exercise 5.2 (Gysin sequence in homology.)
Establish for an orientable (k-1)-sphere bundle $\xi\colon S \to B$ over an orientable manifold B the Gysin sequence in homology:

$$\ldots \longrightarrow H_i(S) \xrightarrow{\xi_*} H_i(B) \xrightarrow{\cap\chi} H_{i-k}(B) \xrightarrow{\sigma_*} H_{i-1}(S) \longrightarrow \ldots$$

where $\chi = \chi_\xi \in H^k(B)$ is the Euler class of ξ.

Werner Gysin (1915 - ?), swiss topologist, student of Heinz Hopf, of whom little is known, not even a picture.

Exercise 5.3 (Cohomology ring of the complex Stiefel manifolds.)
Compute with the Gysin sequence the cohomology rings of the complex Stiefel manifolds:

$$H^*(V_k(\mathbb{C}^n);\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(s_{2n-2k+1}, s_{2n-2k+3} \ldots, s_{2n-1}), \qquad \text{whith } |s_i| = i..$$

In particular, $H^*(U(n);\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(s_1, s_3, \ldots, s_{2n-1})$ for the unitary groups.

Eduard Stiefel (1909 - 1978), swiss topologist and student of Heinz Hopf.

Exercise 5.4 (The pseudo-manifold of a cycle.)
Let $a = \sum_\alpha \lambda_\alpha b_\alpha$ be an integral m-cycle in a space X, where α runs in a finite index set A, the coefficients $\lambda_\alpha \neq 0$ and $b_\alpha\colon \Delta^m \to X$ are basis elements in the chain group $S_m(X)$. In the boundary

$$\partial a = \sum_\alpha (-1)^i \lambda_\alpha \partial_i(a) = \sum_{\alpha,i} (-1)^i\, b_{\alpha,i}$$

we abbreviate the faces by $b_{\alpha,i} := b_\alpha \circ d_i$. Because of $\partial a = 0$ we can choose a 'cancellation pairing' $\pi\colon A \times \{0, \ldots m\} \to A \times \{0, \ldots m\}$ among the faces $b_{\alpha,i}$, which means :

$$(1) \quad b_{\alpha,i} = b_{\beta,j} \quad \text{and} \quad (2) \quad (-1)^i \operatorname{sign}(\lambda_\alpha) + (-1)^j \operatorname{sign}(\lambda_\beta) = 0, \qquad \text{if } \pi(\alpha, i) = (\beta, j).$$

Take an m-simplex Δ_α for each $\alpha \in A$ and define the *cycle space* to be the finite polyhedron

$$P(a) := (\bigsqcup_A \Delta_\alpha)/\sim_\pi$$

where $\sim_\pi$ identifies the faces of the Δ_α according to the pairing π:

$$b_{\alpha,i}(t) \sim_\pi b_{\beta,j}(t) \text{ for all } t \in \Delta^{m-1}, \quad \text{if } \pi(\alpha,i) = (\beta,j).$$

Example*: If $X = M$ is a connected, compact, closed, orientable and triangulated m-manifold and a the fundamental cycle, i.e., the the sum of all m-simplices with appropriate signs (determined by the chosen orientation), then there is an obvious pairing (namely each $(m-1)$-simplex is contained in precisely two m-simplices which we pair). With this pairing $P(a) = M$ and $[a] = u_M$ is the fundamental class determined by the orientation.*

(a) Show by example that $P(a)$ depends on the pairing π.

(b) If $P(a)$ is connected, the homology group $H_m(P(a);\mathbb{Z})$ is infinite cyclic and generated by an obvious *fundamental class* v_a. (For this reason we call $P(a)$ a pseudo-manifold.)

(c) Construct a map $\Phi\colon P(a) \to X$ such that $\Phi_*(v_a) = [a]$ for the induced homomorphism $\Phi_*\colon H_m(P(a);\mathbb{Z}) \to H_m(X;\mathbb{Z})$.

Exercise 5.5* (How to use the cap-product and Poincaré duality to prove the homology and cohomology groups of a compact manifold to be finitely generated.)

Let M be a compact, connected and orientable manifold of dimension m. Assume we are given a space X together with a homology class $v \in H_m(X;\mathbb{Z})$ and a map $f\colon X \to M$ such that $f_*(v) = u_M$, the fundamental class of M. The cap product satisfies

$$f_*(f^*(\alpha) \cap b) = \alpha \cap f_*(b) \quad \text{for } \alpha \in H^*(M) \text{ and } b \in H_*(X).$$

(a) Fixing $b = v$, we have $f_*(f^*(\alpha) \cap v) = \alpha \cap u_M$ and thus the commutative diagram

$$\begin{tikzcd} H^q(X) \arrow[d, "\cap v"'] & H^q(M) \arrow[l, "f^*"'] \arrow[d, "\cap u_M = \mathrm{PD}_M"] \\ H_{m-q}(X) \arrow[r, "f_*"'] & H_{m-q}(M) \end{tikzcd}$$

where PD_M is the Poincaré isomorphism. (Is the left vertical map $\beta \mapsto \beta \cap v$ an isomorphism ?)

(b) Conclude that f^* is injective and f_* is surjective in each degree.

Examples*:*
(1) $f\colon X = M \times N \to M$ and N orientable.
(2) $f\colon X = M \# N \to M$ smashing N minus an open disc to a point, and N orientable.
(3) $f\colon X =$ Klein bottle $\to \mathbb{S}^1$, a non-orientable $\mathbb{S}^1$-bundle over $\mathbb{S}^1$.
(4) $f\colon X = \widetilde{M} \to M$ an r-fold covering, $v = f_!(u_M)$ the fundamental class transfered up; this time one gets on the right r times the Poincaré isomorphism.

(c) Apply this to the cycle space $P(a)$ from Exercise 5.4 for a cycle a representing the fundamental class $[a] = u_M$ of M (and any pairing π). Conclude that $H_*(M;\mathbb{Z})$ is finitely generated in each degree. And all cohomology groups $H^*(M;\mathbb{Z})$ are finitely generated.

(d) For a connected and compact, but non-orientable manifold consider the two-fold orientation cover, use the transfer, the mod-2-Bockstein, and the Universal Coefficient Theorem to show that the homology and cohomology groups are finitely generated.