Exercises for Algebraic Topology I

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Blatt 4 due by: 08.11.2017

Exercise 4.1 (A compact, connected, non-orientable 3-manifold .....)
.... has an infinite fundamental group.

(Hint: Use the Euler charcteristic to show that $H_1(M)$ must be infinite. You can use he fact that the homology groups of compact manifolds is finitely generated in each degree.)

Exercise 4.2 (H_1 of open euclidean sets)
Let $W \subset \mathbb{R}^3$ be an open set. Can $H_1(W)$ have torsion ?
Is the same true for open sets of $\mathbb{R}^2$? Or of $\mathbb{R}^m$ with $m > 3$?

Exercise 4.3 (Manifolds and their boundary)
If M is a compact manifold with non-empty boundary ∂M, can there be a retraction of M onto the boundary ?
(Hint: Consider the long exact homology sequence of the pair $(M, \partial M)$ and show, that $H_{m-1}(\partial M) \to H_{m-1}$ is not injective.)

Exercise 4.4 (Transfer for finite coverings)
For a finite covering $f\colon N = \tilde{M} \to M$ of compact, connected and orientable manifolds of dimension m we can define the homology transfer and cohomology transfers

$$f_!\colon H_q(M) \longrightarrow H_q(N) \quad \text{resp.} \quad f^!\colon H^q(N) \longrightarrow Hq(M)$$

using the Poincare Duality homomorphism. In this case of coverings we can define these transfers even for r-fold coverings of arbitrary spaces $f\colon \tilde{X} \to X$ as follows. For a basis element $a\colon \Delta^q \to X$ in the singular chain group $S_q(\tilde{X})$ we define

$$f_!\colon S_q(X) \longrightarrow S_q(\tilde{X}), \quad f_!(a) := \sum \tilde{a},$$

where the sum is over all lifts $\tilde{a}\colon \Delta^q \to \tilde{X}$ of a, thus $f \circ \tilde{a} = a$. This sum is a finite, since there are only finitely many lifts $\tilde{a}$ of a. Similarily, define

$$f^!\colon S^q(\tilde{X}) \longrightarrow S^q(X), \quad f^!(\beta)(a) := \sum \beta(\tilde{a}),$$

where the sum is again over all lifts $\tilde{a}$ of a.

(a) Show that $f_!$ is a chain map, and $f^!$ is a cochain map.

(b) The compositions $S_\bullet(f) \circ f_!$ and $f^! \circ S^\bullet(f)$ are multiplication by r.

(c) We have induced transfer homomorphisms

$$f_!\colon H_q(X) \longrightarrow H_q(\tilde{X}), \quad \text{and} \quad f^!\colon H^q(\tilde{X}) \longrightarrow H^q(X)$$

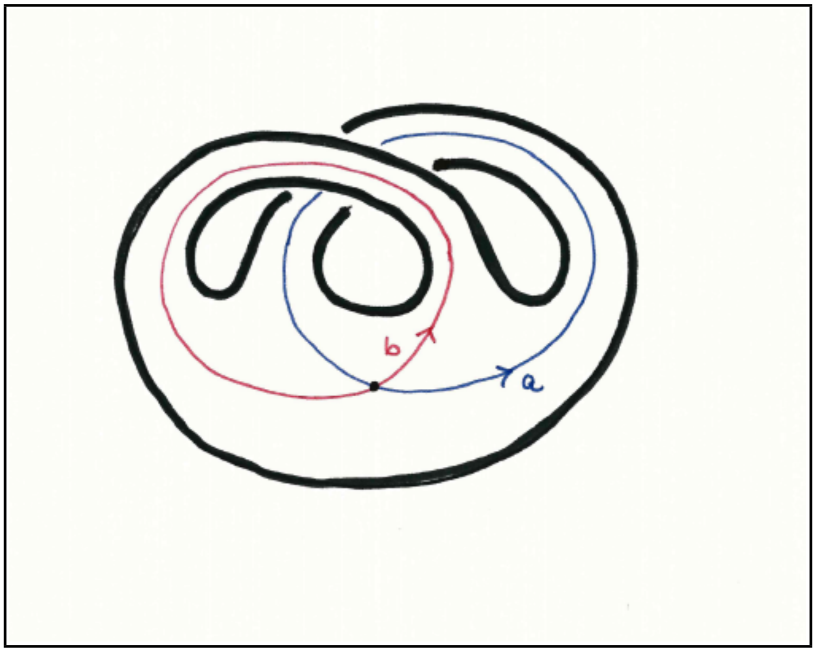
and the compositions $f_* \circ f_!$ and $f^! \circ f^*$ are multiplication by r.

(d) Prove that $H_q(\tilde{X}; \mathbb{Q})$ contains $H_q(X; \mathbb{Q})$ as a direct summand. And $H^q(\tilde{X}; \mathbb{Q})$ contains $H^q(X; \mathbb{Q})$ as a direct summand.

Exercise 4.5 (Intersection product and transfer)
Let a, b be two homology classes in a compact, connected and orienatble m-manifold. Show the following formula, expressing the intersection product via the transfer and the homology cross product:

$$a \bullet b = d_!(a \times b),$$

where $d\colon M \to M \times M$ is the diagonal map.



A Seifert surface of genus one spanning a trivial knot.

Exercise 4.6* (Seifert form of a knot, linking numbers and intersection form)
Let $F \subset \mathbb{R}^3$ be a compact, connected and oriented surface with one boundary curve K. This K can be knotted; in fact, every knot (or even link) in $\mathbb{R}^3$ is the boundary of such a surface, called a *Seifert surface spanning* K; and there are infinitely many surfaces spanning the same K.
On F we have the intersection form on the free group $H_1(F;\mathbb{Z}) \cong \mathbb{Z}^{2g}$

$$I\colon H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}, \quad (a,b) \mapsto I(a,b) \quad \text{with} \quad a \bullet b = I(a,b)\, u_{(F,\partial F)}$$

where g is the genus of F, and $u_{(F,\partial F)}$ denotes the relative fundamental class and generator in $H_2(F,\partial F;\mathbb{Z})$. Note that the intersection form I does of courses not depend on the embedding of F in $\mathbb{R}^3$. We know that I is skew-symmetric in the case of a surface, and non-degenerate.
Next we define the *Seifert form* of the embedded surface

$$S\colon H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}.$$

First, we need to define it only for simly closed curves on F in $\mathbb{R}^3$, since they generate $H_1(F)$. Secondly, we can assume that we have a thickening $F \times [-1,1] \subset \mathbb{R}^3$ such that $F = F \times \{0\}$. We denote for any curve c on F by c^+ the projection of c to the 'upper side' $F^+ = F \times \{+1\}$ and by c^- the projection of c to the 'lower side' $F^- = F \times \{-1\}$. Since for any two simply closed curves a and b on F the curves a^+ and b^- are disjoint curves in $\mathbb{R}^3$, their linking number is defined and we set

$$S(a,b) := \operatorname{link}(a^+, b^-).$$

Note that the Seifert form does depend on the embedding of F into $\mathbb{R}^3$, in contrast to the intersection form I.

(a) Compute intersection form I and the Seifert form S for the knots and Seifert surfaces shown in the two figures above and below.

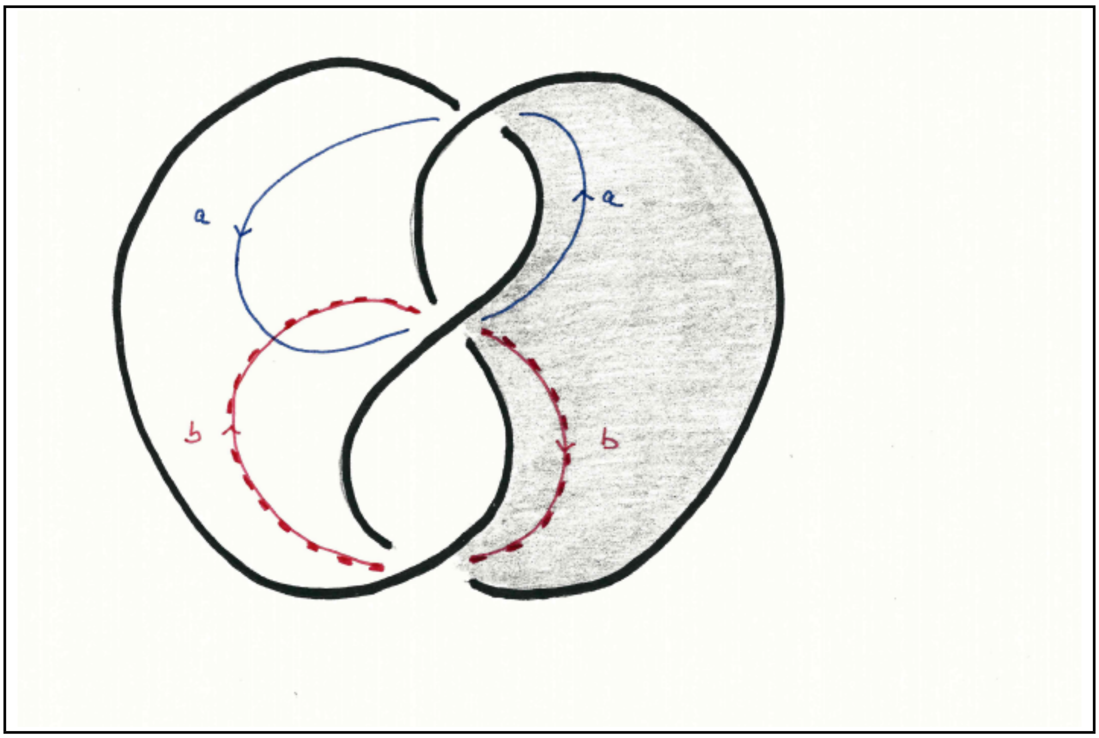
(b) In these two examples the Seifert form is neither symmetric nor skew-symmetric. But show that

$$S(a,b) - S(b,a) = I(a,b) \qquad \text{for all} \quad a,b \in H_1(F).$$

(c) Send the relative fundamental class $u_{(F,\partial F)} \in H_2(F, \partial F; \mathbb{Z})$ of the surface F via the inclusion to $H_2(\mathbb{R}^3, K; \mathbb{Z})$ and denote the image by v. If $J \subset \mathbb{R}^3 - K$ is any (oriented) curve disjoint from K, we regard it as a homology class $[J] \in H_1(\mathbb{R}^3 - K; \mathbb{Z})$. Show that the linking number of K and J can be expressed as a relative intersection number of J and a Seifert surface F spanning K, namely

$$\mathrm{link}(K, J) = \mathrm{PD}^{-1}(v) \cap [J] \in H_0(\mathbb{R}^3 - K) = \mathbb{Z},$$

where PD denotes Poincaré duality.



A Seifert surface of genus one spanning a trefoil knot.