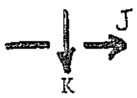
D. LINKING NUMBERS.

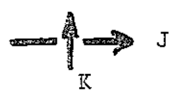
Let J and K be two disjoint oriented knots in S^3 (or R^3). This section describes eight methods for defining an integer called their <u>linking number</u>, all of which turn out to be equivalent, at least up to sign. Assume J and K are polygonal.

(1) Let $[J]$ be the homology class in $H_1(S^3 - K)$ carried by J. Since $H_1(S^3 - K) \cong Z$, we may choose a generator γ of this group and write $[J] = n\gamma$. Define $\ell k_1(J,K) = n$.

(2) Let M be a PL Seifert surface for K, with bicollar (N,N^+,N^-) of $\mathring{M}$ as in the previous section. Assume (allowing adjustment of J by a homotopy in $S^3 - K$) that J meets M in a finite number of points, and at each such point J passes locally (a) from N^- to N^+ or (b) from N^+ to N^-, following its orientation. Weight the intersections of type (a) with $+1$ and those of type (b) with -1. The sum of these numbers we denote by $\ell k_2(J,K)$. [Note that this seems to depend on M].

(3) Consider a regular projection of $J \cup K$. At each point at which J crosses <u>under</u> K, count

$+1$ for 

and -1 for 

The sum of these, over all crossings of J under K, is called $\ell k_3(J,K)$.

(4) J is a loop in $S^3 - K$, hence represents an element of $\pi_1(S^3 - K)$ with suitable basepoint. This fundamental group abelianizes to Z, and the loop J is thereby carried to an integer, called $\ell k_4(J,K)$.

(5) [J] and [K] are 1-cycles in S^3. Choose a 2-chain $C \in C_2(S^3;Z)$ such that $[K] = \partial C$. Then the intersection $C \cdot [J]$ is a 0-cycle, well-defined up to homology. Since $H_o(S^3) \cong Z$, $C \cdot [J]$ corresponds to an integer which we call $\ell k_5(J,K)$.

(6) Regard $J, K : S^1 \to R^3$ as maps.

In vector notation, define a map $f : S^1 \times S^1 \to S^2$ by the formula

$$f(u,v) = \frac{K(u) - J(v)}{|K(u) - J(v)|} .$$

If we orient $S^1 \times S^1$ and S^2 then f has a well-defined degree. Let $\ell k_6(J,K) = \deg(f)$.

(7) (Gauss Integral) Define $\ell k_7(J,K)$ to be the integer

$$\frac{1}{4\pi}\int_J\int_K \frac{(x'-x)(dydz'-dzdy') + (y'-y)(dzdx'-dxdz') + (z'-z)(dxdy'-dydx')}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{3/2}}$$

where (x,y,z) ranges over J and (x',y',z') over K.

(8) Let $p : \tilde{X} \to X$ be the infinite cyclic cover of $X = S^3 - K$ and let τ generate $\mathrm{Aut}(\tilde{X})$. Consider J as a loop in X based at, say, $x \in \mathrm{Im}\, J$. Lift J to a path $\tilde{J}$ in $\tilde{X}$, starting at any $\tilde{x}_o \in p^{-1}(x)$ and call its terminal point $\tilde{x}_1 \in p^{-1}(x)$. There is a unique integer m such that $\tau^m(\tilde{x}_o) = \tilde{x}_1$. Define $\ell k_8(J,K) = m$.

1. EXERCISE. Identify the choice in each of the above definitions which affects the sign of the linking number.

2. THEOREM. $\ell k_i = \pm \ell k_j$; $i, j = 1, \ldots, 8$.

E. BOUNDARY LINKING. Recall that a link $L^n \subset R^{n+2}$ is a <u>boundary link</u> if its components bound disjoint Seifert surfaces. To establish that a link is a boundary link merely requires a construction; to show one is <u>not</u>, may require more cunning. This section discusses some methods, by example, and establishes that both boundary and non-boundary links $L^n \subset R^{n+2}$ exist for all $n \geq 1$. First some crude criteria.

1. PROPOSITION : If any two components of $L^1 \subset S^3$ or R^3 have nonzero linking number, L is <u>not</u> a boundary link.

PROOF : Use definition (2) of linking number.

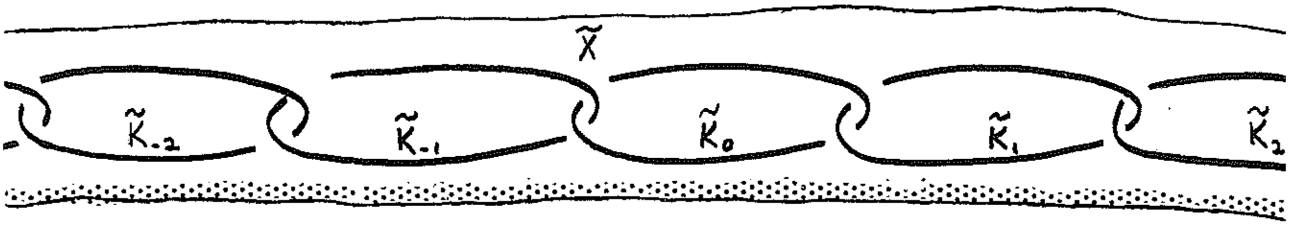
2. PROPOSITION : If $L^n \subset S^{n+2}$ or R^{n+2} is splittable, then L is a boundary link. (Assuming L is PL or C^∞)

PROOF : Assume the components $L_1, \ldots, L_r$ lie interior to disjoint balls $B_1^{n+2}, \ldots, B_r^{n+2}$. Then (EXERCISE) one may construct homeomorphisms $h_i : R^{n+2} \to \text{int } B_i^{n+2}$ which are fixed on L_i. By theorem B1, each L_i bounds a Seifert surface M_i^{n+1}. Then the surfaces $h_i(M_i)$ are again Seifert surfaces for the L_i, and they are disjoint, as required.

3. EXAMPLE : Whitehead's link is not a boundary link. For if M_J and M_K are Seifert surfaces for J and K, respectively, one may construct the universal abelian (= universal) cover

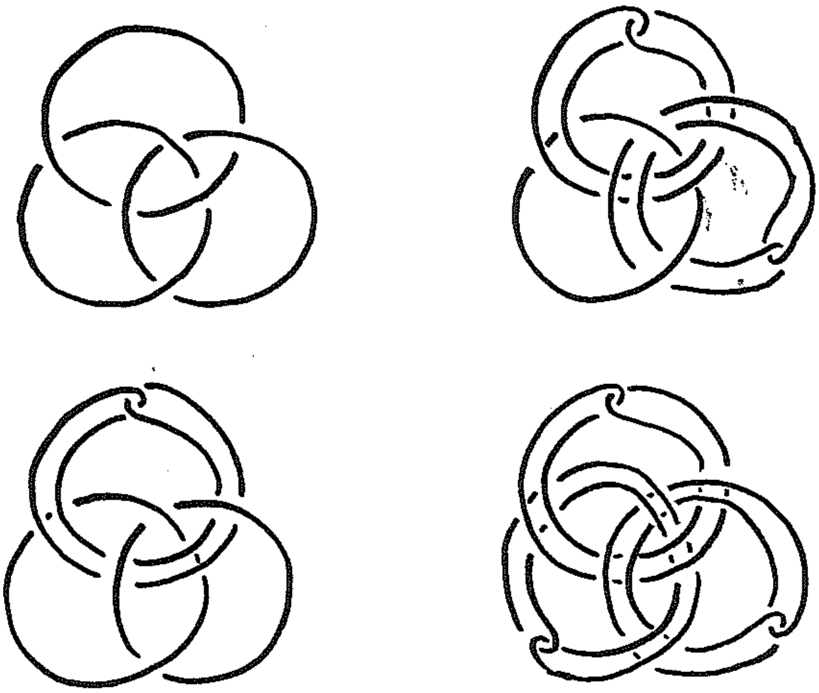
$\widetilde{X}$ of $X = S^3 - J$ by cutting along M_J. If $M_J \cap M_K = \phi$, then M_K

lifts to disjoint (why?) Seifert surfaces ... $\tilde{M}_{-1}$, $\tilde{M}_0$, $\tilde{M}_1$ ... for the liftings ... $\tilde{K}_{-1}$, $\tilde{K}_0$, $\tilde{K}_1$ ... of K in $\tilde{X}$. This is impossible,



since any two consecutive liftings of K have linking number 1 .

That boundary linking is a fairly subtle property is exhibited by the following variations on the Borromean link.



4. EXERCISE : In the above picture, the two links on the right are boundary links. Those on the left are not.