Exercises for Algebraic Topology I

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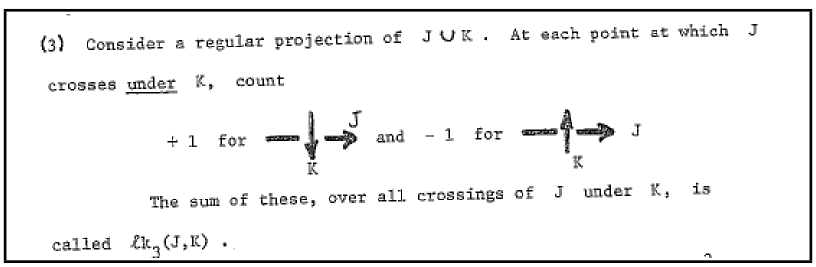
Winter Term 2017/18

Blatt 3 due by: Thursday, 2.11.2017 noon, hand to the secretary Mrs. Müller-Moewes, office 4.011

Aufgabe 3.1 (Linking numbers).
Let M, N be two disjoint, connected, compact and oriented manifolds of dimension p resp. q in euclidean space $\mathbb{R}^m$ and $p+q=m-1$.

(a) Show that $\mathrm{lk}(M,N)=0$ if M and N lie on differnt sides of a hyperplane.

(b) For two curves J,K in $\mathbb{R}^3$ choose a regular projection onto the plane, i.e., one without trible crossings. For each crossing point define a sign $\pm$ according to the figure below. Let $\mathrm{lk}_3(J,K)$ be the sum of these numbers divided by 2. Show that this agrees with our definition (up to perhaps a sign).
(Hint: Here $m=3$ and $p=q=1$, so the homomolgy group in question is $H_1(\mathbb{R}^3-J)$. Use now the Wirtinger presentation of $\pi_1(\mathbb{R}^3-J)$ and thus a presentation for its abelinization $H_1(\mathbb{R}^3-J)$.)


(3) Consider a regular projection of $J \cup K$. At each point at which J crosses under K, count

$+1$ for [figure: J, K] and -1 for [figure: J, K]

The sum of these, over all crossings of J under K, is called $\ell k_3(J,K)$.

From D. Rolfsen: *Knots and Links*, p. 132 - 138.
The book gives eight equivalent definitions for the linking number of two curves in $\mathbb{R}^3$. The first one is the one we use in the lecture. The third one uses a graphical calculus and is used in Exercise 3.1(b).

Aufgabe 3.2 (Cohomology with compact support)

For a locally compact space X one defines the *cohomology with compact support* by

$$H^q_{cpt}(X;\mathbb{A}) := \lim_{K} H^q(X, X-K, \mathbb{A}),$$

where the limit runs over all compact $K \subset X$. Obviously, $H^q_{cpt}(M;\mathbb{A}) \cong H_m(M;\mathbb{A})$ if M is compact.
(a) Compute $H^m_{cpt}(\mathbb{R}^m;\mathbb{Z}) \cong \mathbb{Z}$.
(b) Compute $H^m_{cpt}(M;\mathbb{Z}) \cong \mathbb{Z}$ for any orientable m-manifold, and $H^m_{cpt}(M,\mathbb{Z}/2) \cong \mathbb{Z}/2$ for any orientable m-manifold, compact or not.

Let the coefficients be $\mathbb{Z}$ and assume M is orientable, i.e. there is a coherent family $o=(o_p \mid p \in M)$ of generators in

(6) Regard $J, K : S^1 \to R^3$ as maps.

In vector notation, define a map $f : S^1 \times S^1 \to S^2$ by the formula

$$f(u,v) = \frac{K(u) - J(v)}{|K(u) - J(v)|} .$$

If we orient $S^1 \times S^1$ and S^2 then f has a well-defined degree. Let $\ell k_6(J,K) = \deg(f)$.

Definition 6 is by the degree of the (normalized) distance map from a torus $\mathbb{S}^1 \times \mathbb{S}^1$ to the 2-sphere $\mathbb{S}^2$. It has an astronomical interpretation: if an astronomer on a star S_J moving along the curve J measures the direction towards a star S_K moving along the curve K for the times of two entire revolutions of S_J and S_K, he measures all directions, if and only if the two curves are linked.

(7) (Gauss Integral) Define $\ell k_7(J,K)$ to be the integer

$$\frac{1}{4\pi}\int_J\int_K \frac{(x'-x)(dydz'-dzdy') + (y'-y)(dzdx'-dxdz') + (z'-z)(dxdy'-dydx')}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{3/2}}$$

where (x,y,z) ranges over J and (x',y',z') over K.

This Definition 7 is the oldest, given by Gauss as the integral: He used to measure the current in a wire K induced by a current in a wire J.

$H_m(M, M-p)$. If M is not compact, we may not have a fundamental class in $H_m(M)$, see for example $M = \mathbb{R}^m$. But we do have a fundamental class $u_K \in H_m(M, M-K)$ for any compact $K \subset M$, thus an orientation or fundamental class along K. We can cap cohomology classes with this u_K and obtain a direct system homomorphisms

$$\mathrm{PD}_K \colon H^q(M, M-K) \longrightarrow H_{m-q}(M), \quad \alpha \mapsto \alpha \cap u_k,$$

we go to the direct limit and obtain a dualtiy homomorphism

$$\mathrm{PD}_{cpt} \colon H^q_{cpt}(M) \longrightarrow H_{m-q}(M).$$

Show, that it is an isomorphism for any orientable M.

Aufgabe 3.3 (Transfer)
Prove one the following formulas for the homology or cohomology transfer of a map $f \colon N \to M$ of connected, compact, closed and orientable manifolds.

(1) $f_!(\alpha \cap a) = f^*(\alpha) \cap f_!(a)$ for $\alpha \in H^*(M)$ and $a \in H_*(M)$.

(2) $f_*(\beta \cap f_!(a)) = f^!(\beta) \cap a$ for $\beta \in H^*(N)$ and $a \in H_*(M)$.

(3) $f^!(f^*(\alpha) \cup \beta) = \alpha \cup f^!(\beta)$ for $\alpha \in H^*(M)$ and $\beta \in H^*(N)$.

Consider now the case of a projection map $f = p_M \colon N = M \times F \to M$, where M and F are connected, compact, closed and orientable manifolds of dimension m and r, so $n = m+r$ is the dimension of N. Show that the cohomology transfer

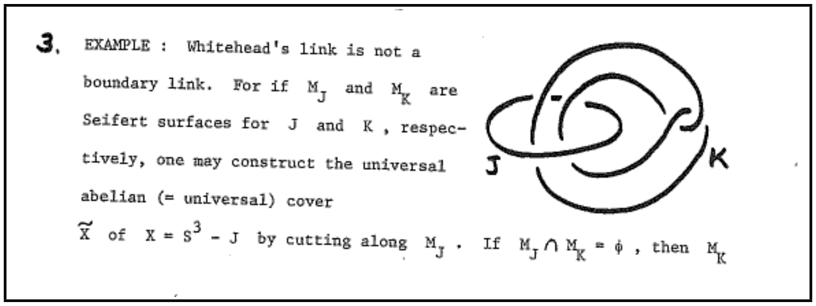
$$f^! \colon H^{n-p}(M \times F) \to H^{m-p}(M)$$

is trivial on cohomology cross products $\alpha \times \beta$ with $\alpha \in H^*(M)$ and $\beta \in H^*(F)$ unless $|\beta| = r$; in this non-trivial case we have $f^!(\alpha \times \omega_F) = \alpha$, when ω_F is the cohomology fundamental class of F.

Consider the homology thansfer

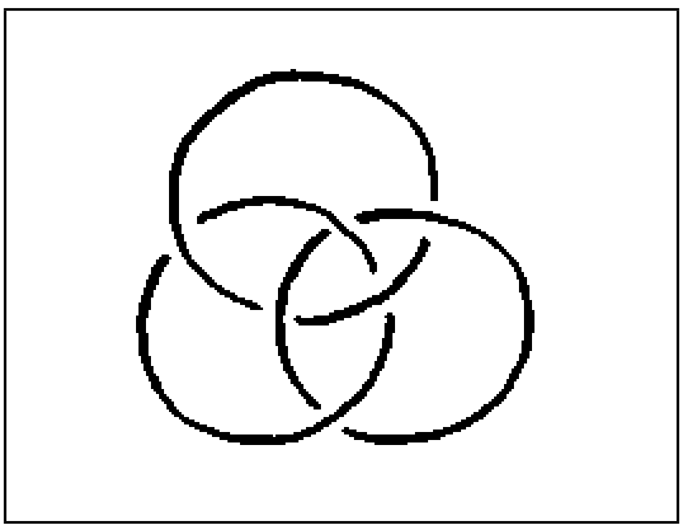
$$f_! \colon H_{m-p}(M) \to H_{n-p}(M \times F)$$

and show that $f_!(a) = a \times u_F$, when u_F is the homology fundamental class of F.



3. EXAMPLE : Whitehead's link is not a boundary link. For if M_J and M_K are Seifert surfaces for J and K, respectively, one may construct the universal abelian (= universal) cover $\widetilde{X}$ of $X = S^3 - J$ by cutting along M_J. If $M_J \cap M_K = \phi$, then M_K

This Whitehead link can not be 'unlinked', although the linking number is zero.



For the three Borromean rings the linking number of any two is zero; any two could be separated, but not in the complement of the third.

Aufgabe 3.4 (Not null-bordant manifolds.)
The even projective spaces $\mathbb{R}P^{2n}$ and $\mathbb{C}P^{2n}$ are not the boundary of any connected, compact and oriented manifold.