Exercises for Algebraic Topology I

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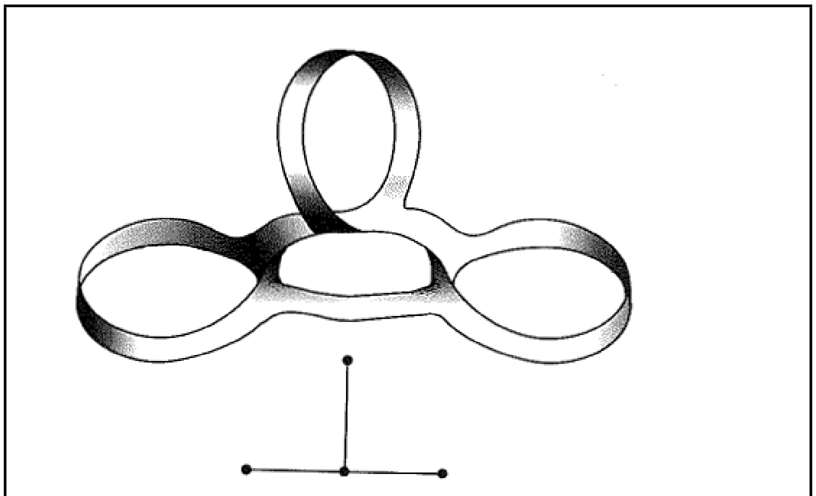
Winter Term 2017/18

Blatt 2 due by: 25.10.2017

Exercise 2.1 (Euler characteristic)
For a connected, compact, closed and orientable manifold of odd dimension the Euler characteristic must be zero.



from A.A.Kosinski: Differentiable Manifolds, p. 121; Plumbing and homology spheres.

Exercise 2.2 (Maps to projective spaces)
(a) Assume $f\colon X \to \mathbb{R}P^n$ is a map from a connected space X with $H^k(X;\mathbb{Z}/2) = 0$ for some $0 < k < n$. Then the induced homomorphism $f^*\colon H^n(\mathbb{R}P^n;\mathbb{Z}/2) \to H^n(X;\mathbb{Z}/2)$ is zero.

(b) Assume $f\colon X \to \mathbb{C}P^n$ is a map from a connected space X with $H^{2k}(X;\mathbb{Z}) = 0$ for some $0 < k < n$. Then the induced homomorphism $f^*\colon H^{2n}(\mathbb{C}P^n;\mathbb{Z}) \to H^{2n}(X;\mathbb{Z})$ is zero.

Exercise 2.3 (Homology slant product)
Let $\mathbb{K}$ be commutative ring with a unit 1. The *homology slant product*

$$/\colon H^{p+q}(X \times Y;\mathbb{K}) \otimes H_q(Y;\mathbb{K}) \longrightarrow H^p(X;\mathbb{K}), \quad \gamma \otimes b \mapsto \gamma/b$$

is defined as follows. First consider for two chain complexes A and B the slant pairing

$$/\colon \operatorname{Hom}_{\mathbb{K}}(A \otimes_{\mathbb{K}} B, \mathbb{K}) \otimes_{\mathbb{K}} B \longrightarrow \operatorname{Hom}_{\mathbb{K}}(A, \mathbb{K}), \quad \gamma \otimes b \mapsto \gamma/b$$

where $(\gamma/b)(a) := \gamma(a \otimes b)$ for $\gamma\colon A \otimes B \to \mathbb{K}$ and $a \in A_p$, $b \in B_q$. The formula

$$\delta(\gamma/b) = \delta(\gamma)/b - (-1)^{p+q}\, \gamma/\partial(b)$$

is immediate and tells us, that this product is well-defined on cohomology classes and homology classes:

$$/ \colon H^{p+q}(A \otimes B) \otimes_{\mathbb{K}} H_q(B) \longrightarrow H^p(A).$$

For two spaces X and Y we take their singular chain resp. cochain complexes $A = S^\bullet(X)$, $B = S_\bullet(Y)$ and $C = S^\bullet(X \times Y)$ and recall the Eilenberg-Zilber transformation $\mathrm{EZ} \colon S_\bullet(X) \otimes S_\bullet(Y) \to S_\bullet(X \times Y)$ and its dual $\mathrm{ez} := \mathrm{Hom}_{\mathbb{K}}(\mathrm{EZ}, \mathrm{id}).\colon C = S^\bullet(X \times Y) \to S^\bullet(X) \otimes S^\bullet(Y) = A \otimes B$. Then we set

$$\gamma/b := \mathrm{ez}(\gamma)/b, \qquad \text{for } \gamma \in S^{p+q}(X \times Y), b \in S_q(Y),$$

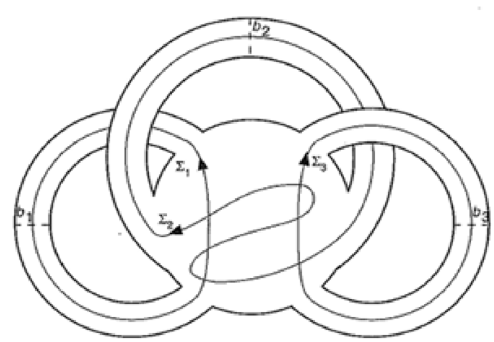
which is a cochain of degree p. Passing to cohomology resp. homology classes gives the desired homology slant product.

Prove two of the following three formulas:

(1) (naturality) $\;(f \times g)^*(\gamma)/b' = f^*(\gamma/g_*(b'))\quad$ for $f \colon X' \to X$, $g \colon Y' \to Y$, $\gamma \in H^{p+q}(X \times Y)$ and $b' \in H_q(Y')$.

(2) (cohomology cross product) $\;(\alpha \times \beta)/b = \langle \beta, b \rangle\, \alpha\quad$ for $\alpha \in H^p(X)$, $\beta \in H^q(Y)$ and $b \in H_q(Y)$.

(3) (homology cross product) $\;\langle \gamma/b, a \rangle = \langle \gamma, a \times b \rangle\quad$ for $\gamma \in H^{p+q}(X \times Y)$, $a \in H_p(X)$ and $b \in H_q(Y)$.



$$\mathfrak{I} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Figure VI,11

(12.2) Proposition *∂B is a homotopy sphere if and only if the intersection matrix $\mathfrak{I}$ is unimodular.* $\square$

loc. cit., p. 119.

Exercise 2.4 (Diagonal class of a manifold)
Let M be a connected, compact, closed and oriented m-manifold; the coefficients of homology and cohomology will be in a field $\mathbb{F}$.

If $u \in H_m(M)$ denotes the fundamental class, then the cross product $u \times u$ is a fundamental class for $M \times M$. With the diagonal map $d\colon M \to M \times M$ we get the *homology diagonal class* $u_\Delta := d_*(u) \in H_m(M \times M)$. Its Poincaré dual $\omega_\Delta \in H^m(M \times M)$ we call the *cohomology diagonal class*. So we note $\omega_\Delta \cap (u \times u) = u_\Delta$.

Now let $\beta_i \in H^*(M)$ be a basis of the cohomology and let $\beta_i' \in H^*(M)$ be the dual basis in the sense that $\langle \beta_i \cup \beta_j', u\rangle = 1$ for $i = j$ and $= 0$ for $i \neq j$. Show that

$$\omega_\Delta = \sum_i (-1)^{|\beta_i|}\, \beta_i' \times \beta_i$$

expresses the cohomology diagonal class in terms of the cohomology cross product.

Exercise 2.5 (Poincaré duality, once more)
Let M be a connected, compact and oriented m-manifold; let the coefficients of homology and cohomology be in the field $\mathbb{F}$. Using the homology slant product we can define the inverse Poincaré duality homomorphism:

$$\mathrm{pd}\colon H_{m-q}(M) \longrightarrow H^q(M), \quad b \mapsto \omega_\Delta / b,$$

where ω_Δ denotes the cohomology diagonal class. Show that pd is an isomorphism and inverse to PD.

$$\mathrm{PD}\colon H^q(M) \longrightarrow H_{m-q}(M), \quad \alpha \mapsto \alpha \cap u,$$

where $u \in H_m(M)$ denotes the fundamental class.

Exercise 2.6* (Hopf invariant)
For a map $f\colon \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ let $C(f)$ denote its mapping cone $C(f) := \mathbb{S}^{2n} \cup_f e^{4n}$, a 2n-sphere with a 4n-cell attached via f. First compute the integral cohomology:

(1) $H^{2n}(C(f)) \cong \mathbb{Z}$

(2) $H^{4n}(C(f)) \cong \mathbb{Z}$

(3) $H^i(C(f)) = 0$ for $i \neq 0, 2n, 4n$

Choose generators $\alpha \in H^{2n}(C(f))$ and $\beta \in H^{4n}(C(f))$ and define the the natural number $\mathbf{h}(f)$ by the equation

$$\alpha^2 = \mathbf{h}(f)\, \beta.$$

It is called the *Hopf invariant* of the map f. Thus $H^*(C(f))$ is the quotient ring of $\mathbb{Z}[\alpha, \beta]$ with generators in degrees 2 and 4 modulo the ideal generated by $\alpha^2 - \mathbf{h}(f)\beta$.
Prove the following statements.

(4) $\mathbf{h}(f \circ \phi) = \deg(\phi)\, \mathbf{h}(f)$ for any self-map $\phi\colon \mathbb{S}^{4n-1} \to \mathbb{S}^{4n-1}$;

(5) $\mathbf{h}(\psi \circ f) = \deg(\psi)^2\, \mathbf{h}(f)$ for any self-map $\psi\colon \mathbb{S}^{2n} \to \mathbb{S}^{2n}$;

(6)* $\mathbf{h}(f_1 + f_2) = \mathbf{h}(f_1) + \mathbf{h}(f_2)$, for any two maps $f_1, f_2\colon \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$,

where the sum $f_1 + f_2 = F \circ (f_1 \vee f_2) \circ T$ is the composition of the bouquet $f_1 \vee f_2$ with the map $T\colon \mathbb{S}^{4n-1} \to \mathbb{S}^{4n-1} \vee \mathbb{S}^{4n-1}$ squeezing the equator to a point, and the fold map $F\colon \mathbb{S}^{2n} \vee \mathbb{S}^{2n} \to \mathbb{S}^{2n}$.

Finally, can you find for some n a map $f\colon \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ with Hopf invariant $\mathbf{h}(f) = 3$?

The problem of finding a map $f\colon \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ with Hopf invariant $\mathbf{h}(f) = 1$ is equivalent to the famous problem of finding real division algebras. And this problem is equivalent to the parallelizability of $\mathbb{R}P^{2n-1}$ and of $\mathbb{S}^{2n-1}$. In 1960 J. Adams showed it is only possible for $n = 1, 2$ and 4, where one did know examples. — Read the article by F. Hirzebruch: Divisionsalgebren und Topologie in the wonderful book Zahlen, edited by H.-D. Ebbinghaus et al. .