Exercises for Algebraic Topology I

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Blatt 1 due by: 18.10.2017

Aufgabe 1.1 (Poincaré Duality mod 2).
For a connected, compact and closed m-manifold M we define a *fundamental class mod 2* to be a non-zero class in $H_m(M;\mathbb{Z}/2)$ such that the 'restriction' $o_p := \rho_{p,M}(u)$ in $H_m(M, M-p;\mathbb{Z}/2) \cong \mathbb{Z}/2$ is non-zero (und thus a generator) for each point $p \in M$.

(a) Show that <u>any</u> M as above has such a class and it is unique.

(b) If M is orientable, then the mod-2-reduction of any fundametal class in $H_m(M;\mathbb{Z})$ is <u>the</u> fundamental class mod 2.

(c) Define the mod-2 Poincare homomorphism by

$$PD\colon H^q(M;\mathbb{Z}/2) \to H_{m-q}(M;\mathbb{Z}/2), \quad PD(\alpha) := \alpha \cap u$$

and show it is an isomorphism for compact M.

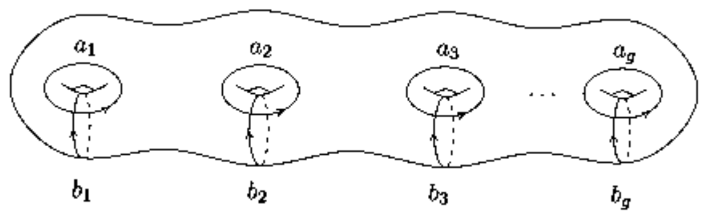

Figure 6.1 The standard geometric symplectic basis for $H_1(S_g;\mathbb{Z})$.

B. Farb, D. Margalit: A Primer on Mapping Class Groups, page 165.
For any homeomorphism $f\colon S_g \to S_g$ the induced homomorphism on $H_1(S_g) \cong \mathbb{Z}^{2g}$ determines a matrix A_f in $\mathrm{GL}_{2g}(\mathbb{Z})$ with respect to the basis in the figure. If f is orientation-preserving, A_f is a matrix in $\mathrm{SL}_{2g}(\mathbb{Z})$. Moreover, since f_* preserves the intersection form (see Aufg. 1.2(c)), the matrix A_f is even in the symplectic subgroup $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Aufgabe 1.2 (Intersection form and signature)
Let $m = 2n$ be even and consider a compact, connected and orientable manifold of dimension m. with its intersection form

$$I\colon H_n(M;\mathbb{Z}) \times H_n(M;\mathbb{Z}) \to \mathbb{Z}, \quad \text{defined by the equation} \quad a \bullet b = I(a,b)[x]$$

where $\bullet$ denotes the intersection product and where $[x]$ is the ground class in $H_0(M)$.
Show that

(a) $I(a,b) = (-1)^n I(b,a)$.

(b) $I(a,b) = 0$, if a or b is of finite order.

(c) $I(f_*(a), f_*(b)) = I(a,b)$ for any orientation-preserving self-map of S_g.

Compute the intersection form and the signature of the following manifolds:

(0) $M = S_g$, an orientable surface of genus g; see figure above.

(1) $M = \mathbb{S}^n \times \mathbb{S}^n$.

(2) $M = \mathbb{S}^{m_1} \times \cdots \times \mathbb{S}^{m_k}$ with $m = m_1 + \ldots m_k$ even.

(3) $M = \mathbb{C}P^n$.

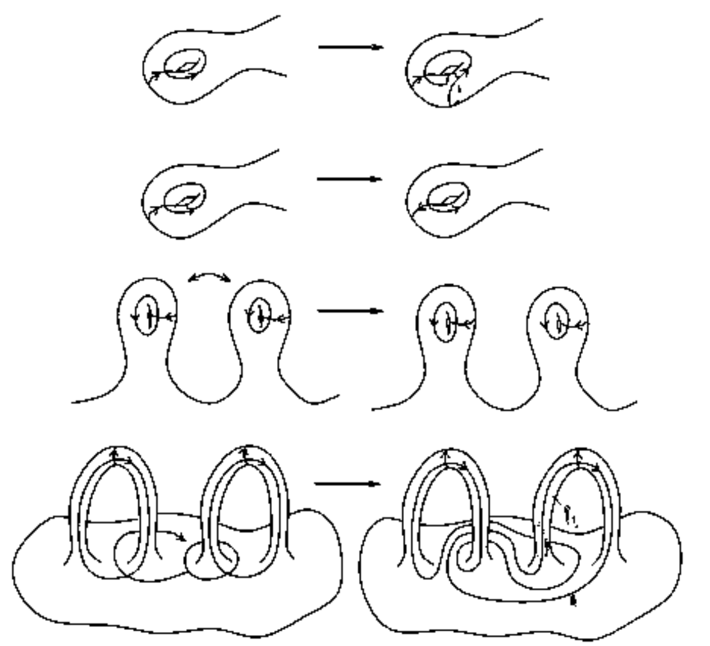

Figure 6.3 **Realizing the Burkhardt generators geometrically. From top to bottom: a transvection, a factor rotation, a factor swap, and a factor mix.**

loc. cit. , p.age 171. — Denote by $\Gamma_g = \pi_0(\mathrm{Homeo}^+(S_g))$ the group of homotopy classes of orientation-preserving homeomorphisms of the surface S_g. It is called the mapping class group. As remarked above, we have a homomorphism $\Gamma_g \to \mathrm{Sp}_{2g}(\mathbb{Z})$ to the symplectic group. The drawings amount to a proof, that it is an epimorphism, since the transvections generate the symplectic group.

Aufgabe 1.3 (Connected sum of two manifolds)
For two closed manifolds M_1 and M_2 of the same dimension m recall the definition of their connected sum $M = M_1 \# M_2$ as the union of M_1 and M_2 with a disc removed in each of them and identified along the boundaries of these discs. The result is a new manifold, which is connected resp. compact if both M_1 and M_2 are so; it is orientable, if both M_1 and M_2 are so and if we identify the boundaries with an orientation-preserving homeomorphism.

(a) Compute the homology groups of M in terms of the homology groups of M_1 and M_2.

(b) Now assume that $m = 2n$ and M_1 and M_2 are both orientable. Using $H_n(M) \cong H_n(M_1) \oplus H_n(M_2)$, show

$$I_M((a_1, a_2), (b_1, b_2)) = I_{M_1}(a_1, b_1) + I_{M_2}(a_2, b_2)$$

for the intersection form of M.

Aufgabe 1.4 (Free parts and torsion parts of homology and cohomology)
For a connected, compact, closed and orientable manifold M show with the help of Poincaré Duality and the Universal Coefficient Theorem:

- $\mathrm{free}(H^q(M)) \cong \mathrm{free}(H_q(M)) \cong \mathrm{free}(H_{m-q}(M))$
- $\mathrm{tors}(H^q(M)) \cong \mathrm{tors}(H_{q-1}(M)) \cong \mathrm{tors}(H_{m-q}(M))$

Here $\mathrm{free}(A)$ denotes the free part and $\mathrm{tors}(A)$ the torsion part of a finitely-generated abelian group A. (We use that these homology and cohomology groups are finitely-generated.)
Can one refine the second statement by considering only the torsion with respect to a certain prime ?

Aufgabe 1.5* (Degrees of self-maps)
For a connected, oriented, compact and closed m-manifold M we can use the fact $H_m(M) \cong \mathbb{Z}$ to define a *degree* $\deg(f) \in \mathbb{Z}$ for any self-map $f\colon M \to M$ by the equation

$$f_*(u) = \deg(f)\, u,$$

where u is a fundamental class.

(1) With the other fundamental class $-u$ we get the same degree.

(2) Clearly, $\deg(f) = 0$ for a null-homotopic map.

(3) And $\deg(f) = \pm 1$ for a homotopy-equivalence.

(4) Furthermore, $\deg(f \circ g) = \deg(f)\, \deg(g)$ for the composition of two self-maps.

(5) For $M = \mathbb{S}^m$ we know that $\deg\colon [\mathbb{S}^m, \mathbb{S}^m] \to \mathbb{Z}$ is a bijection; in particular, each natural number is the degree of some self-map. Surprisingly, this is not true in general; for $M = \mathbb{C}P^n$ the degree of any self-map must be an n-th power.
Hint: Look at the top cohomology group instead, which is generated by α^n with $\alpha \in H^2(\mathbb{C}P^n)$ a generator.