Harmonic Analysis, Problem set 11

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Winter term 2016/17

Due on Thursday, 2017-01-19

Definition. An *filter* p on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ with the following properties:

(a) $\emptyset \notin p$, $\mathbb{N} \in p$.

(b) If $A, B \in p$, then $A \cap B \in p$,

(c) If $A \subseteq B \subseteq \mathbb{N}$ and $A \in p$, then $B \in p$.

Let X be a topological space and $f : \mathbb{N} \to X$ a function. A limit of f along a filter p is a point $x \in X$ such that for every neighborhood U of x we have $f^{-1}(U) \in p$. This is denoted by $\lim_{n\to p} f(n) = x$.
An *ultrafilter* is a filter that is maximal with respect to inclusion. An ultrafilter is called *principal* if if has the form $\{A : n \in A\}$ for some $n \in \mathbb{N}$.

Problem 1. In this problem you will show that there exist multiplicative linear functionals on $\ell^\infty(\mathbb{N})$ that are not given by evaluation at a point.

(a) Show that every filter is contained in an ultrafilter. Show that there exists a non-principal ultrafilter.

(b) Let p be an ultrafilter and $A \cup B \in p$. Show that $A \in p$ or $B \in p$.

(c) Let $f : \mathbb{N} \to [-1, 1]$ be a function and p an ultrafilter. Show that $\lim_{n\to p} f(n)$ exists and is unique.

(d) Let $f, g : \mathbb{N} \to [-1, 1]$ and p an ultrafilter. Show that $(\lim_{n\to p} f(n))(\lim_{n\to p} g(n)) = \lim_{n\to p} f(n)g(n)$. Hence the ultralimit defines a multiplicative linear functional on $\ell^\infty(N)$ (you do not have to verify linearity).

(e) Show that if $f \mapsto \lim_{n\to p} f(n)$ coincides with the point evaluation at $m \in \mathbb{N}$, then p is the principal ultrafilter at m.

Definition. The *BMO (for "bounded mean oscillation") norm* of a (measurable) function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\|f\|_{\mathrm{BMO}} := \sup_I \inf_{c\in\mathbb{R}} |I|^{-1} \int_I |f - c|,$$

where the supremum is taken over all subintervals of $\mathbb{R}$. The *dyadic BMO norm* is defined similarly with a supremum over dyadic intervals I.
The space of functions with finite BMO (resp. dyadic BMO) norm is denoted by BMO (resp. BMO_d)

Problem 2. (a) Show that $\|f\|_{\mathrm{BMO}} \le \|f\|_\infty$

(b) Show that the function $\log|x|$ is in BMO.

(c) Show that the function $1_{x>0} \log|x|$ is in BMO_d, but not in BMO.

(d) Show that

$$\|f\|_{\mathrm{BMO}} \le \sup_I |I|^{-1} \int_I |f - f_I| \le 2\|f\|_{\mathrm{BMO}}, \quad f_I = |I|^{-1} \int_I f.$$

(e) Prove the (dyadic) *John–Nirenberg inequality*: there exist constants $C, c > 0$ such that for every dyadic interval I

$$|I \cap \{f - f_I > \lambda\}| \le C \exp(-c\lambda/\|f\|_{\mathrm{BMO}_d})|I|, \quad 0 \le \lambda < \infty$$

Hint: one can assume $\|f\|_{\mathrm{BMO}_d} = 1$ and $f_I = 0$, and it suffices to consider $\lambda = 10N$, $N \in \mathbb{N}$. Construct inductively a sequence of subsets $I_N \subset I$ starting with $I_0 = I$ as follows: each $I_N = \cup_i I_{N,i}$ will be a disjoint union of dyadic intervals. Given I_N define

$$I_{N+1} := \cup_i (I_{N,i} \cap \{M_d(f - f_{I_{N,i}}) > 5\}),$$

where M_d is the dyadic Hardy–Littlewood maximal function. By induction on N show:

(e.1) $f \le 10N$ on $I \setminus I_N$,
(e.2) $|I_N| \le \exp(-cN)|I|$,
(e.3) $|f_{I_{N,i}}| \le 10N$.