Exercise 1

a) Let $c_0 = \{x \in l^\infty(\mathbb{N};\mathbb{C}) : \lim_{j\to\infty} x_j = 0\}$. Show: c_0 is a closed subspace of $l^\infty(\mathbb{N})$.

b) Show: $(c_0)^*$ is isomorphic to $l^1(\mathbb{N},\mathbb{C})$, i.e. there exists a bijective isometry.

c) Show or falsify: c_0^* is isomorphic to $(l^\infty)^*$.

Exercise 2

a) Let X be a normed space. Show: Every proper subspace $V \subset X$, $V \neq X$ has empty interior.

b) Let P be the vector space of all real polynomials. Use a) to prove that there exists no norm $\|.\|$ such that $(P, \|.\|)$ is a Banach space.

Exercise 3

a) Let X be a complex normed space and $S \subset X$ an arbitrary subset. Show: If $\{x^*(x) : x \in S\}$ is a bounded set in $\mathbb{C}$ for each $x^* \in X^*$ then there exists K such that $\|x\| \leq K$ for all $x \in S$.

b) Use a) to show: If the norms $\|.\|_1$ and $\|.\|_2$ on a vector space V are not equivalent, then there exists $x^* : V \to \mathbb{C}$ linear which is continuous with respect to one of the norms but not the other.

Exercise 4

a) Find a sequence of functions (f_j) such that $f_j \rightharpoonup 0$ in $L^2(\mathbb{R})$ but (f_j) is not a Cauchy sequence in $L^2(\mathbb{R})$.

b) Prove or falsify

   (a) If $f_j \rightharpoonup f, g_j \to g$ in $L^4(\mathbb{R})$ then $f_j g_j \rightharpoonup fg$ in $L^2(\mathbb{R})$.

   (b) If $f_j \rightharpoonup f, g_j \rightharpoonup g$ in $L^4(\mathbb{R})$ then $f_j g_j \rightharpoonup fg$ in $L^2(\mathbb{R})$.

   (c) If $f_j \rightharpoonup f, f_j \to g$ in $L^2(\mathbb{R})$ then $f = g$.

Exercise 5

Let $(H, \langle ., . \rangle)$ be a real Hilbert space.

a) Let (x_j) be a sequence in H. Prove $x_n \rightharpoonup x^* \quad \iff \lim_{j\to\infty} \langle x_j, h\rangle = \langle x^*, h\rangle$ for all $h \in H$.

b) Let (x_j) be a sequence in H and $x \in H$. Prove: $x_j \to x \quad \in H \quad \iff x_j \rightharpoonup x, \|x_j\| \to \|x\|$.

Hint: For $\Longleftarrow$ you might want to expand $\|x_j - x\|^2$.

Exercise 6

Let $I = (0,1) \subset \mathbb{R}$, $b > 0$ and $f \in L^2(I)$ be a given function. Prove that there exists exactly one $u \in W^{2,2}(I)$ such that

$$B(u,\phi) = \int_0^1 u''\phi'' + bu'\phi' + u\phi dx = \int_0^1 f\phi dx$$

for all $\phi \in W^{2,2}(I)$.

Exercise 7

Show that there exists a continuous linear functional $L \in (C_b([0,\infty)))^*$ such that $L(f) = \lim_{x\to\infty} f(x)$ whenever the limit exists.

Exercise 8

a) Consider $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$, $(Tx)_k = \frac{1}{k+1}x_k$. Show that T is a compact operator.

b) Consider $T : L^2([0,1]) \to L^2([0,1])$ given by $(Tf)(x) = \frac{1}{x+1}f(x)$ is bounded but not compact.

Exercise 9

Let $p \in (1,\infty)$, $U \subset \mathbb{R}^d$ open and bounded. Let $f_j, f \in L^p(U)$, $j \in \mathbb{N}$. Show

$$f_j \rightharpoonup f \text{ in } L^p(U) \quad \Longleftrightarrow \quad \sup_j \|f_j\|_{L^p(U)} < \infty, \quad \text{and} \quad f_j \to f \text{ in } \mathcal{D}'(U).$$

Exercise 10

Let $U_1, U_2 \subset \mathbb{R}^d$ be open and bounded. Let $K \in C(\overline{U_1 \times U_2})$. For $g \in C(\overline{U_1})$ let $Tf(x) = \int_{U_2} K(x,y)f(y)dm^d(y)$.

a) Prove that $T \in L(C(\overline{U_2}), C(\overline{U_1})$.

b) $\|T\|_{C(\overline{U_2})\to C(\overline{U_1}))} = \sup_{x\in U_1} \int_{U_2} |K(x,y)|dm^d(y)$

c) Prove that T is compact.

Exercise 11

Let $U \subset \mathbb{R}^d$ be open and bounded.

a) Prove that if $u \in W^{1,2}(U)$, then $u^+ = \max\{u,0\} \in W^{1,2}(U)$, $u^- = \max\{-u,0\} \in W^{1,2}(U)$ and $|u| \in W^{1,2}(U)$. Compute their derivative.

b) Let $u \in W^{1,2}(U)$ satisfy $\int \sum_{j=1}^d \partial_j u \partial_j v dm^d \le 0$ for all $v \in W_0^{1,2}(U)$ with $v \ge 0$ almost everywhere and $u^+ \in W_0^{1,2}(U)$. Prove that $u \le 0$ almost everywhere.

Exercise 12

a) Let X be a Banach space, let $x_j, x \in X$ for $j \in N$. Prove that $x_n \rightharpoonup x$ implies $\|x\| \leq \liminf \|x_j\|$.

b) Find $f_j, g_j \in L^\infty(0,1)$ for $j \in \mathbb{N}$ such that $f_j \rightharpoonup^* f$, $g_j \rightharpoonup^* g$ but $f_j g_j$ does not weak* converge to fg.

Exercise 13

Let X be a Hilbert space and $x \in X$. Let (e_k) be an orthonormal system.

a) Show: For each $N \in \mathbb{N}$ $\sum_{k=1}^N |\langle x, e_k \rangle|^2 \leq \|x\|^2$.

b) Prove or falsify: $e_k \rightharpoonup 0$.