Notes on

Functional Analysis and Partial Differential Equations

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These are short incomplete notes. They do not substitute textbooks. The following textbooks are recommended.

- H. W. Alt, Linear functional analysis, Springer, 2016.
- D. Werner, Funktionalanalysis, Springer, 2011.

Correction are welcome and should be sent to `koch@math.uni-bonn.de` or told me during office hours. The notes are only for participants of the course V3B1/F4B1 *PDG und Funktionalanalysis /PDE and Functional Analysis* at the University of Bonn, Winter term 2016/2017. A current version can be found at

`http://www.math.uni-bonn.de/ag/ana/WiSe1617/V3B1_WS_16.html`.

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1 Introduction

Functional analysis is the study of normed complete vector spaces (called Banach spaces) and linear operators between them. It is built on the structure of linear algebra and analysis. Functional analysis provides the natural frame work for vast areas of mathematics including probability, partial differential equations and numerical analysis. It expresses an important shift of viewpoint: Functions are now points in a function space.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. One of the deepest results in *Einführung in die PDG* was that the Green's function $g(x, y)$ provides a map

$$f \to u$$

given by

$$u(x) = \int g(x,y)f(y)dy =: Tf$$

so that

$$-\Delta u = f \qquad \text{in } \Omega$$
$$u = 0 \qquad \text{on } \partial\Omega$$

whenever f is sufficiently regular. It is not hard to see that

$$T : L^2(\Omega) \to L^2(\Omega)$$

and T is one of the most relevant operators in functional analysis.

The main abstract objects are topological vector spaces over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We will focus on normed spaces, the most important class of topological vector spaces.

Definition 1.1. *Let X be a $\mathbb{K}$ vector space. A map $\|.\| : X \to [0, \infty)$ is called norm if*

$$\|x\| = 0 \qquad \Longleftrightarrow \qquad x = 0, \tag{1.1}$$

if for all $x, y \in X$

$$\|x + y\| \le \|x\| + \|y\| \qquad \textit{(triangle inequality)}\ , \tag{1.2}$$

and if for all $\lambda \in \mathbb{K}$ and $x \in X$

$$\|\lambda x\| = |\lambda|\|x\|. \tag{1.3}$$

It is called normed space and Banach space if it is complete as metric space.

Remark 1.2. *A norm defines a metric by* $d(x,y) = \|x-y\|$.

First examples.

1. $\mathbb{R}^n$ and $\mathbb{C}^n$ equipped with the Euclidean norm are real resp. complex Banach spaces.
2. Let X be a set. The space of bounded functions $\mathbb{B}(X)$ equipped with the supremum norm is a Banach space.
3. Let (X,d) be a metric spaces. The space of bounded continuous functions $C_b(X)$ equipped with the supremum norm is a Banach space, or more precisely a closed sub vector space of $\mathbb{B}(X)$.
4. Let $U \subset \mathbb{R}^d$ be open. $C_b^k(U)$ is the vector space of k times differentiable functions on U which are together with there derivatives bounded. The norm
$$\|u\|_{C^k(U)} = \max_{|\alpha|\le k} \|\partial^\alpha u\|_{\mathbb{B}(U)}$$
turns C_b^k into a Banach space. *Exercise*
5. Let $U \subset \mathbb{C}$ be open. The space of bounded holomorphic functions $H^\infty(U)$ is a Banach space when equipped with the supremum norm.

Lemma 1.3. *Suppose that X is a Banach space and $U \subset X$ is a vector space which is a closed subset of X. Then U is a Banach space.*

Definition 1.4. *Let X and Y be normed spaces. We define $L(X,Y)$ as the set of all continuous linear maps from X to Y.*

Theorem 1.5. *Let $T : X \to Y$ be in $L(X,Y)$. Then*
$$\|T\|_{X\to Y} := \sup_{\|x\|_X \le 1} \|T(x)\|_Y < \infty$$
and $\|.\|_{X\to Y}$ defines a norm on $L(X,Y)$. A linear operator $T : X \to Y$ is continuous if and only if its norm $\|T\|_{X\to Y}$ is finite. $L(X,Y)$ is a Banach space if Y is a Banach space.

Proof. Continuous linear maps from X to Y are a vector space with the obvious sum and multiplication. Let $T : X \to Y$ be a continous linear map. We choose $\varepsilon = 1$ and $x_0 = 0$. Then there exists $\delta > 0$ so that
$$\|Tx\|_Y \le 1 \qquad \text{if } \|x\|_X \le \delta,$$
and hence, if $x \in X$, $x \neq 0$, then $\left\| \frac{\delta x}{\|x\|_X} \right\|_X \le \delta$ and
$$\|Tx\|_Y = \frac{\|x\|_X}{\delta} \left\| T \frac{\delta x}{\|x\|_X} \right\|_Y \le \delta^{-1} \|x\|_X$$

and thus

$$\|T\|_{X\to Y} \le \delta^{-1}.$$

Vice versa: Let $T : X \to Y$ be linear so that $\|T\|_{X\to Y} < \infty$. For $\varepsilon > 0$ we choose $\delta = \varepsilon/\|T\|_{X\to Y}$. Then

$$\|Tx - Ty\|_Y = \|T(x-y)\|_Y \le \|T\|_{X\to Y}\|x-y\|_X \le \varepsilon$$

provided $\|x-y\|_X \le \delta$. In particular T is uniformly continuous.

Now assume that Y is a Banach space and let $T_n \in L(X,Y)$ be a Cauchy sequence. For all x, $T_n x$ is a Cauchy sequence in Y since

$$\|T_n x - T_m x\|_Y \le \|T_n - T_m\|_{X\to Y}\|x\|_X.$$

Let

$$Tx := \lim_{n\to\infty} T_n x.$$

The convergence is uniform on bounded sets, and hence the limit T is continuous and in $L(X,Y)$. Moreover

$$\begin{aligned}\|T - T_n\|_{X\to Y} = \sup_{\|x\|_X\le 1} \|(T-T_n)x\|_Y = \sup_{\|x\|_X \le 1} \limsup_{m\to\infty} \|(T_m - T_n)x\|_Y \\ \le \limsup_{m\to\infty} \|T_m - T_n\|_{X\to Y} \to 0\end{aligned}$$

as $n \to \infty$. Here we used continuity of addition and the map to the norm. $\square$

Definition 1.6 (Dual space)**.** *Let X be a normed space. We define the dual (Banach) space as $X^* = L(X,\mathbb{K})$.*

Example: Let $X = \mathbb{R}^n$ with the Euclidean norm. The map

$$\mathbb{R}^n \ni y \to (x \to \sum_{j=1}^n x_j y_j) \in (\mathbb{R}^n)^*$$

is isometric and surjective. It allows to identify $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$.

[19.10.2016]

[21.10.2016]

Lemma 1.7. *The space $C_0(\mathbb{R}^n) \subset C_b(\mathbb{R}^n)$ of functions converging to 0 at ∞ is a Banach space. Similarly the space c_0 of sequences converging to 0 equipped with the sup norm is a Banach space.*

Lemma 1.8. *Let X be a normed space. Addition, scalar multiplication and the map to the norm are continuous.*

Lemma 1.9. *The closure of a subvector space of a normed spaces is a subvector space.*

Further examples: Let $1 \le p \le q \le \infty$. We define the sequence spaces

Definition 1.10. *A $\mathbb{K}$ sequence $(x_j)_{j\in\mathbb{N}}$ is called p summable if*

$$\sum_{j=1}^{\infty} |x_j|^p < \infty \text{ if } p < \infty, \qquad \sup_{j=1}^{\infty} |x_j| < \infty \quad \text{ if } p = \infty.$$

The set of all p summable sequences is denoted by $l^p(\mathbb{N}) = l^p$.

Theorem 1.11. *The set of p summable sequences is a vector space. The expressions*

$$\|(x_j)\|_{l^p} = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p}, \quad p < \infty$$

resp.

$$\|(x_j)\|_{l^\infty} = \sup_{j\in\mathbb{N}} |x_j|$$

are norms on $l^p(\mathbb{N})$, which turn $l^p(\mathbb{N})$ into Banach spaces. If $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p, q \le \infty$ and $(x_j) \in l^p$, $(y_j) \in l^q$ then $(x_j y_j)$ is summable and Hölder's inequality holds:

$$\left| \sum_{j=1}^{\infty} x_j y_j \right| \le \sum_{j=1}^{\infty} |x_j y_j| \le \|(x_j)\|_{l^p} \|(y_j)\|_{l^q}.$$

Remark 1.12. *We may replace $\mathbb{N}$ by $\mathbb{Z}$, by a finite set, or even an arbitrary set. Then $l^\infty(X) = \mathbb{B}(X)$. The triangle inequality is called Minkowski inequality.*

We recall Young's inequality

$$|xy| \le \frac{1}{p}|x|^p + \frac{1}{q}|y|^q$$

for $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$ and $x, y \in \mathbb{R}$. Without loss of generality we assume $x, y > 0$ and this can be proven by searching the maximum of

$$x \to xy - \frac{1}{p}x^p$$

for $y > 0$ which is attained at $x_0 = y^{1/(p-1)}$:

$$x_0 y - \frac{1}{p}x_0^p = \frac{p-1}{p} y^{\frac{p}{p-1}} = \frac{1}{q} y^q.$$

As a consequence

$$\sum_j |x_j y_j| \le \sum_j \frac{1}{p}|x_j|^p + \frac{1}{q}|y_j|^q = \frac{1}{p}\|(x_j)\|_{l^p}^p + \frac{1}{q}\|(y_j)\|_{l^q}^q$$

and we obtain Hölder's inequality

$$\begin{aligned}\sum_j |x_j y_j| =& \|(x_k)\|_{l^p}\|(y_k)\|_{l^q}\sum_j |x_j|\|(x_k)\|_{l^p}^{-1}|y_j|\|(y_k)\|_{l^q}^{-1}\\ =& \|(x_k)\|_{l^p}\|(y_k)\|_{l^q}(\frac{1}{p}+\frac{1}{q})\\ =& \|(x_j)\|_{l^p}\|(y_j)\|_{l^q}.\end{aligned}$$

Proof. Since $l^\infty(\mathbb{N}) = \mathbb{B}(\mathbb{N})$ there is nothing to show if $p = \infty$. Moreover the triangle inequality is obvious if $p = 1$. Then, if $1 < p, q < \infty$

$$\begin{aligned}\sum_{j=1}^\infty |x_j + y_j|^p \le& \sum_{j=1}^\infty |x_j + y_j|^{p-1}|x_j + y_j|\\ \le& \sum_{j=1}^\infty |x_j + y_j|^{p-1}|x_j| + \sum_{j=1}^\infty |x_j + y_j|^{p-1}|y_j|\\ \le& \|(|x_j + y_j|^{p-1})\|_{l^q}\Big(\|(x_j)\|_{l^p} + \|(y_j)\|_{l^p}\Big)\\ =& \|(x_j + y_j)\|_{l^p}^{p-1}\Big(\|(x)_j\|_{l^p} + \|(y_j)\|_{l^p}\Big)\end{aligned}$$

and

$$\|(x_j + y_j)\|_{l^p} \le \|(x_j)\|_{l^p} + \|(y_j)\|_{l^p}.$$

provided we can devide by $\|(x_j + y_j)\|_{l^p}$. There is nothing to show if this quantity is 0 and it is finite whenever we sum over a finite number of indices. Then a limit argument gives the full statement.

In particular we obtain the triangle inequality. One easily sees that $\|(x_j)\|_{l^p} = 0$ implies $(x_j) = 0$ and

$$\|(\lambda x_j)\|_{l^p} = |\lambda|\|(x_j)\|_{l^p}.$$

Thus the spaces l^p are normed vector spaces. Now suppose that $x_n = (x_{n,j})$ is a Cauchy sequence in l^p. Then for every j, $n \to x_{n,j}$ is a Cauchy sequence in $\mathbb{K}$. Let $y_j = \lim_{n\to\infty} x_{n,j}$ and $y = (y_j)$. Then, for every $m > 1$ (assuming $p < \infty$, since $l^\infty = \mathbb{B}(\mathbb{N})$)

$$\begin{aligned}\|y - x_m\|_{l^p}^p =& \lim_{N\to\infty}\sum_{j=1}^N |y_j - x_{m,j}|^p\\ =& \lim_{N\to\infty}\lim_{n\to\infty}\sum_{j=1}^N |x_{n,j} - x_{m,j}|^p\\ \le& \lim_{n\to\infty}\|x_n - x_m\|_{l^p}^p\\ \to& \quad 0 \qquad \text{as } m \to \infty.\end{aligned}$$

$\square$

Definition 1.13. *Two norms $\|.\|$ and $|.|$ on a normed spaces X are called equivalent, if there exits $C \ge 1$ so that for all $x \in X$*

$$C^{-1}\|x\| \le |x| \le C\|x\|.$$

Theorem 1.14. *All norms on finite dimensional spaces are equivalent. Finite dimensional normed spaces are Banach spaces.*

Proof. Let $|.|$ be the Euclidean norm on $\mathbb{K}^d$ and $\|.\|$ a second norm. Let $\{e_j\}_{j=1,\cdots,d}$ be the standard basis. Then

$$\Big\| \sum_{j=1}^d a_j e_j \Big\| \le \sum_{j=1}^d |a_j| \max_k \|e_k\| \le (\sqrt{d} \max_k \|e_k\|) \left| \sum_{j=1}^d a_j e_j \right|.$$

Thus $v \to \|.\|$ is continuous with respect to $|.|$. It attains the infimum on the Euclidean unit sphere (which is compact). This minimum has to be positive and we call it λ^{-1}. Then

$$|v| = |v||v/|v|| \le \lambda |v| \|v/|v|\| = \lambda \|v\|.$$

The two inequalities imply the equivalence of the norms $\|.\|$ and $|.|$ by choosing

$$C = \max\left\{\sqrt{d} \max \|e_k\|, \lambda, 1\right\}.$$

Thus every norm on $\mathbb{K}^d$ is equivalent to the Euclidean norm, and any two norms are equivalent.

A Cauchy sequence $v_m = (v_{m,j})$ with respect to $\|.\|$ is also a Cauchy sequence with respect to the Euclidean norm, hence it converges to a vector v with respect to the Euclidean norm, and hence also $\|v_m - v\| \to 0$.

This proves the claim for $\mathbb{K}^d$. Now let X be a vector spaces of dimension d. Then there is a basis of d vectors, and a bijective linear map ϕ from $\mathbb{K}^d$ to X. If $\|.\|_X$ is a norm on X then $x \to \|\phi(x)\|_X$ is a norm on $\mathbb{K}^d$. Thus the first part follows. Since $\phi(x_n)$ is a Cauchy sequence with respect to $\|.\|_X$ iff (x_n) is a Cauchy sequence in $\mathbb{K}^d$ with respect to the second metric, and one converges iff the second converges.This completes the proof.

□

[21.10.2016]

[26.10.2016]

Lemma 1.15. *Let X be a Banach space and U be a closed subvector space. Then X/U is a vector space,*

$$\|\tilde{x}\|_{X/U} = \inf_{y \in U} \|y - x\|$$

defines a norm (here $\tilde{x}$ is the equivalence class of x) and X/U is a Banach space.

Proof. Exercise □

Lemma 1.16. *Let X and Y be normed spaces. Their direct sum $X \oplus Y (= X \times Y)$ is a vector space. If $1 \leq p \leq \infty$ then*

$$\|(x, y)\|_p = \|(|x|_X, |y|_Y)\|_{l^p}$$

defines a norm with which $X \oplus Y$ becomes a Banach space.

Proof. Exercise □

2 Hilbert spaces

2.1 Definition and first properties

Definition 2.1. *Let X be a $\mathbb{K}$ vector space. A map $\langle ., . \rangle : X \times X \to \mathbb{K}$ is called inner product if*

$$\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle \qquad \text{for all } x_1, x_2, y \in X \tag{2.1}$$

$$\langle \lambda x, y \rangle = \lambda \langle x, y \rangle \qquad \text{for all } x, y \in X, \lambda \in \mathbb{K} \tag{2.2}$$

$$\langle x, y \rangle = \overline{\langle y, x \rangle} \qquad \text{for all } x, y \in X \tag{2.3}$$

In particular $\langle x, x \rangle \in \mathbb{R}$ for all $x \in X$ and

$$\langle x, x \rangle \geq 0 \qquad \text{for all } x \in X \tag{2.4}$$

$$\langle x, x \rangle = 0 \qquad \iff \qquad x = 0 \tag{2.5}$$

Examples:

1. Euclidean vector spaces over $\mathbb{K}$, Euclidean inner product.
2. Real and complex square summable sequences space $l^2(\mathbb{N})$ with $\langle (x_j), (y_j) \rangle = \sum x_j \overline{y_j}$.
3. Let $U \subset \mathbb{R}^n$ be measurable, $X = C_b(U)$, $\langle f, g \rangle = \int_U f \overline{g} dm^n$ where m^n denotes the Lebesgue measure.

Lemma 2.2 (Cauchy-Schwarz)**.** *Let X be a vector space with inner product. Then*

$$|\langle x, y \rangle| \leq (\langle x, x \rangle \langle y, y \rangle)^{\frac{1}{2}}$$

for all $x, y \in X$.

Proof. Let $x, y \in X$ and $\lambda \in \mathbb{K}$. Then

$$\begin{aligned} 0 \leq & \langle x - \lambda y, x - \lambda y \rangle \\ = & \langle x, x \rangle - \lambda \langle y, x \rangle - \bar{\lambda} \langle x, y \rangle + |\lambda|^2 \langle y, y \rangle. \end{aligned}$$

If $y = 0$ there is nothing to show, so we assume $y \neq 0$ and define $\lambda = \frac{\langle x,y \rangle}{\langle y,y \rangle}$. Then

$$0 \leq \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle}$$

which implies the Cauchy-Schwarz inequality. $\square$

Lemma 2.3. *The map*

$$x \to \|x\| := \sqrt{\langle x, x \rangle}$$

defines a norm.

With this notation the Cauchy-Schwarz inequality becomes

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

Proof. Clearly $\|x\| \geq 0$, $\|x\| = 0$ iff $x = 0$. Moreover

$$\|\lambda x\|^2 = |\lambda|^2 \|x\|^2$$

and by the Cauchy-Schwarz inequality

$$\begin{aligned} \|x + y\|^2 = & \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2 \\ \leq & \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 \\ = & (\|x\| + \|y\|)^2. \end{aligned}$$

$\square$

Definition 2.4. *A vector space X with an inner product is called pre-Hilbert space. It is a Hilbert space if it is a Banach space.*

Lemma 2.5. *The inner product defines a continuous map from $X \times X$ to $\mathbb{K}$.*

Proof. Exercise $\square$

It is not hard to verify the parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{2.6}$$

for $x, y \in H$ some pre-Hilbert space.

Theorem 2.6 (Jordan von Neumann)**.** *Let X be a normed $\mathbb{K}$ vector space with norm $\|.\|$. We assume that the norm satisfies the parallelogram identity*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{2.7}$$

Then

$$\langle x, y \rangle = \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2\Big) \tag{2.8}$$

if $\mathbb{K} = \mathbb{R}$ and

$$\langle x, y \rangle = \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\Big) \tag{2.9}$$

otherwise defines an inner product such that the norm is the norm of the preHilbert space. Vice versa: The norm of a prehilbert space defines the parallelogram identity.

As a consequence we could define a Hilbert spaces as a Banach space whose norm satisfies the paralellogram identity. By an abuse of notation we call a normed space pre-Hilbert space if it satisfies the parallelogram identity.

Proof. We begin with a real normed spaces whose norm satisfies the parallelogram identity. We define

$$\langle x, y \rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2).$$

Then

$$\langle x, y \rangle = \langle y, x \rangle.$$

Since by the parallelogram identity

$$2\|x+z\|^2 + 2\|y\|^2 = \|x+y+z\|^2 + \|x-y+z\|^2$$

hence

$$\begin{aligned}\|x+y+z\|^2 =& 2\|x+z\|^2 + 2\|y\|^2 - \|x-y+z\|^2 \\ =& 2\|y+z\|^2 + 2\|x\|^2 - \|y-x+z\|^2\end{aligned}$$

and

$$\|x+y+z\|^2 = \|x\|^2 + \|y\|^2 + \|x+z\|^2 + \|y+z\|^2 - \frac{1}{2}\|x-y+z\|^2 - \frac{1}{2}\|y-x+z\|^2$$

$$\|x+y-z\|^2 = \|x\|^2 + \|y\|^2 + \|x-z\|^2 + \|y-z\|^2 - \frac{1}{2}\|x-y-z\|^2 - \frac{1}{2}\|y-x-z\|^2$$

and we arrive at

$$\begin{aligned}\langle x+y,z\rangle =&\frac{1}{4}(\|x+y+z\|^2-\|x+y-z\|^2)\\ =&\frac{1}{4}(\|x+z\|^2-\|x-z\|^2)+\frac{1}{4}(\|y+z\|^2-\|y-z\|^2)\\ =&\langle x,z\rangle+\langle y,z\rangle.\end{aligned}$$

We claim

$$\langle \lambda x,y\rangle=\lambda\langle x,y\rangle$$

for all $x,y\in X$ and $\lambda\in\mathbb{R}$. It obviously holds for $\lambda=1$ by checking the definition, and for all $\lambda\in\mathbb{N}$ be the previous step, hence for all $\lambda\in\mathbb{Z}$. But then it holds for all rational λ and by continuity for $\lambda\in\mathbb{R}$.

We complete the proof for complex Hilbert spaces: We define

$$\langle x,y\rangle=\frac{1}{4}\sum_{k=0}^{3} i^k\|x+i^k y\|^2$$

and observe that $\langle ix,y\rangle=i\langle x,y\rangle$, $\langle x,y\rangle=\overline{\langle y,x\rangle}$ by definition, $\mathrm{Re}\langle x,y\rangle$ is the previous real inner product and $\mathrm{Im}\langle x,y\rangle=\mathrm{Re}\langle x,iy\rangle$. □

Corollary 2.7. *A normed space is a pre-Hilbert space if and only if all two dimensional subspaces are pre-Hilbert spaces.*

Proof. It is a pre-Hilbert space if and only if its norm satisfies the parallelogram identity which holds if and only if the parallelogram identity holds for all two dimensional subspaces. □

This has geometric consequences.

Lemma 2.8. *Let H be a Hilbert space, $K\subset H$ compact, and $C\subset H$ closed and convex, C and K disjoint. Then there exist $x\in K$ and $y\in C$ so that*

$$\|x-y\|=d(C,K)$$

Proof. Let $x_j\in K$ and $y_j\in C$ be a minimizing sequence. Since K is compact there is a subsequence which we denote again by (x_j,y_j) and $x\in K$ so that $x_j\to x$. By the triangle inequality

$$\|x-y_j\|\to d(C,K).$$

Then

$$\begin{aligned}\|y_n-y_m\|^2 =&\|(x-y_n)-(x-y_m)\|^2\\ =&2\|x-y_n\|^2+2\|x-y_m\|^2-\|2x-(y_n+y_m)\|^2\\ \le&2(\|x-y_n\|^2+\|x-y_m\|^2)-4d^2(C,K)\\ &\to 0 \qquad \text{as } n,m\to\infty\end{aligned}$$

since by convexity $\frac{1}{2}(y_n + y_m) \in K$. Thus (y_n) is a Cauchy sequence with limit $y \in C$. Moreover

$$d(C, K) = \lim_{n \to \infty} \|x - y_n\| = \|x - y\|.$$

$\square$

[26.10.2016]

[28.10.2016]

Definition 2.9. *We call two elements $x, y \in H$ orthogonal if $\langle x, y \rangle = 0$.*

Lemma 2.10. *Suppose that C is a closed and convex subset of a Hilbert space H and $x \in H$. Then the closest point in C to x is unique and we denote it by $p(x)$. Moreover, if $y = p(x)$ then*

$$\operatorname{Re}\langle x - y, z - y \rangle \leq 0 \tag{2.10}$$

for all $z \in C$. If $y \in C$ satisfies this inequality for all $z \in C$ then $y = p(x)$. If C is a closed subspace then for all $z \in C$

$$\langle x - p(x), z \rangle = 0. \tag{2.11}$$

The point $y = p(x) \in C$ is uniquely determined by this orthogonality condition. Moreover

$$\|x\|^2 = \|x - p(x)\|^2 + \|p(x)\|^2.$$

Proof. Uniqueness is a consequence of the proof of Lemma 2.8. Let $y \in C$. Then by the triangle inequality, if $z \in C$ then for $0 \leq t \leq 1$

$$y(t) = y + t(z - y) \in C$$

and hence if $y = p(x)$,

$$\|x - y\|^2 \leq \|x - y - t(z - y)\|^2 = \|x - y\|^2 - 2t \operatorname{Re}\langle x - y, z - y \rangle + t^2 \|y\|^2.$$

This implies (2.10) and also the converse. In the case that C is a closed subspace (2.10) is equivalent to the orthogonality relation. $\square$

2.2 The Riesz representation theorem

Theorem 2.11 (Riesz representation theorem)**.** *Let H be a Hilbert space. Then*

$$J : H \ni x \to (y \to \langle y, x \rangle) \in H^*$$

is an $\mathbb{R}$ linear isometric isomorphism. It is conjugate linear:

$$J(\lambda x) = \overline{\lambda} J(x).$$

Proof. By the Cauchy-Schwarz inequality

$$\|J(x)\|_{H^*} = \sup_{\|y\|\le 1} |\langle y, x\rangle| \le \|x\|_H$$

and the map is well defined and antilinear. Since

$$\|x\|_H \|J(x)\|_{H^*} \ge \langle x, J(x)\rangle = \langle x, x\rangle = \|x\|_H^2$$

we see that

$$\|J(x)\|_{H^*} \ge \|x\|_H.$$

Thus J is an isometry: $\|J(x)\|_{H^*} = \|x\|_H$. In particular J is injective. To show that J is surjective we assume that $x^* \in H^*$ and try to find x so that $x^* = J(x)$. Let

$$N = \{y \in H : x^*(y) = 0\}.$$

Then N is a subvector space and it is closed. Let p be the orthogonal projection to N as above. We choose $y_0 \in H$ with $x^*(y_0) = 1$ and define

$$x_0 = y_0 - p(y_0).$$

Then $x^*(x_0) = 1$ and for all $y \in N$ by (2.11) $\langle y, x_0\rangle = 0$. Moreover $x^*(x - x^*(x)x_0) = 0$ hence $x - x^*(x)x_0 \in N$ and by (2.11)

$$\langle x - x^*(x)x_0, x_0\rangle = 0.$$

Since $x = [x - x^*(x)x_0] + x^*(x)x_0$, then

$$\langle x, x_0\rangle = \langle x^*(x)x_0, x_0\rangle = x^*(x)\|x_0\|_H^2$$

and

$$x^*(x) = \left\langle x, \frac{x_0}{\|x_0\|^2}\right\rangle = J(\frac{x_0}{\|x_0\|^2})(x).$$

□

Theorem 2.12 (Lax-Milgram)**.** *Let H be a Hilbert space and*

$$Q : H \times H \ni (x, y) \to Q(x, y) \in \mathbb{K}$$

be linear in x, antilinear in y, bounded in the sense that

$$|Q(x, y)| \le C\|x\|\|y\|$$

and coercive in the sense that there exists $\delta > 0$ so that

$$\operatorname{Re} Q(x, x) \ge \delta\|x\|^2.$$

Then there exists a unique continuous linear map $A : H \to H$ with continuous inverse A^{-1} so that

$$Q(x, y) = \langle Ax, y\rangle.$$

Moreover

$$\|A\|_{H\to H} \le C, \qquad \|A^{-1}\|_{H\to H} \le \delta^{-1}.$$

Proof. Let $x \in H$. Then

$$y \to \overline{Q(x,y)} \in H^*.$$

By the Riesz representation theorem there exists a unique $z(x) \in H$ so that

$$\overline{\langle z(x), y\rangle} = \overline{Q(x,y)}$$

for all $y \in H$. Then

$$\|z(x)\| = \sup_{\|y\|\le 1} |\langle z(x), y\rangle| = \left| \sup_{\|y\|\le 1} Q(x,y) \right|.$$

Clearly $z(x_1 + x_2) = z(x_1) + z(x_2)$ and $z(\lambda x) = \lambda z(x)$ and we define the continuous linear operator $Ax = z$. Since

$$\operatorname{Re}\langle Ax, x\rangle \ge \delta \|x\|^2$$

we obtain

$$\|Ax\| \ge \delta \|x\|.$$

It particular A is injective and the range is closed. If it is not surjective there exists z with $\|z\| = 1$ and z is orthogonal to the range, i.e.

$$\langle Ax, z\rangle = 0$$

for all $x \in X$. In particular we reach the contradiction

$$0 = \langle Az, z\rangle \ge \delta \|z\|^2.$$

□

2.3 Orthonormal systems

We recall that N elements x_j of a vector space X are called linearly independent if

$$\sum_{j=1}^{N} \lambda_j x_j = 0$$

implies $\lambda_j = 0$. Let x_j be N linearly independent vectors of a Hilbert space. By the Gram-Schmidt procedure we obtain an orthonormal system with

$$y_1 = \frac{1}{\|x_1\|} x_1$$

$$\tilde{y}_2 = x_2 - \langle x_1, y_1\rangle y_1$$

$$y_2 = \frac{1}{\|\tilde{y}_2\|} \tilde{y}_2$$

recursively. We can do this with $N = \infty$.

Examples: Orthonormal polynomials. Let $\mu : \mathbb{R} \to (0, \infty)$ be measurable so that all moments exists, i.e.

$$\int_{\mathbb{R}} |x|^N \mu dx < \infty$$

for all $N \geq 0$. Let

$$X = \{f \in C(\mathbb{R}) : (1 + |x|)^{-N} |f| \text{ is bounded for all } N\}.$$

Then

$$X \times X \ni (f, g) \to \langle f, g \rangle := \int f \overline{g} \mu dx$$

defines an inner product. The monomials

$$f_n = x^n$$

are linearly independent.

We consider the case

$$\mu(x) = \begin{cases} 1/2 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

with $X = C([-1, 1])$. It leads to (multiples) of the Legendre polynomials.

Definition 2.13 (Legendre-polynomial). *The Legendre polynomial P_n is the unique polynomial of degree n with $P_n(1) = 1$ and*

$$\int_{-1}^{1} x^m P_n(x) dx = 0$$

for all $0 \leq m < n$.

There is a very compact formula for them.

Lemma 2.14 (Rodrigues formula).

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]$$

Proof. The degree of P_n defined by the right hand side is obviously n and the leading term of $P_n(x)$ reads $\frac{(2n)!}{2^n (n!)^2} x^n$. If $0 \leq m < n$ then after m integrations by parts

$$\int_{-1}^{1} x^m \frac{d^n}{dx^n} (x^2 - 1)^n dx = (-1)^m m! \int_{-1}^{1} \frac{d^{n-m}}{dx^{n-m}} (x^2 - 1)^n dx = 0.$$

Finally

$$\frac{d^n}{dx^n} (x^2 - 1)^n \Big|_{x=1} = 2^n \frac{d^n}{dx^n} (x - 1)^n \Big|_{x=1} = 2^n n!.$$

$\square$

A more difficult calculation gives

$$\begin{aligned}\int_{-1}^{1}(P_n(x))^2\,dx =&\frac{1}{2^n n!}\frac{(2n)!}{2^n(n!)^2}\int_{-1}^{1}x^n\frac{d^n}{dx^n}(x^2-1)^n dx\\ =&\frac{(2n)!}{2^{2n}(n!)^2}(-1)^n\int_{-1}^{1}(x^2-1)^n dx\\ =&\frac{(2n)!}{2^{2n}(n!)^2}2\cdot 2^{2n}\int_0^1 s^n(1-s)^n ds=\frac{2}{2n+1}.\end{aligned}$$

and

$$\sqrt{\frac{2n+1}{2}}\sqrt{\frac{2m+1}{2}}\int_{-1}^{1}P_n(x)P_m(x)\,dx=\delta_{n,m}$$

and the functions

$$\sqrt{\frac{2n+1}{2}}P_n(x)$$

are an orthonormal system.

[28.10.2016]

[02.11.2016]

We consider the Banach space $C([0,2\pi];\mathbb{C})$ with inner product

$$\langle f,g\rangle=\frac{1}{2\pi}\int f\overline{g}dx.$$

It is not hard to see that this is an inner product.

Lemma 2.15. *The functions e^{inx} are an orthonormal system.*

Proof. We compute

$$\frac{1}{2\pi}\int_0^{2\pi}e^{inx}\overline{e^{-imx}}dx=\frac{1}{2\pi}\int_0^{2\pi}e^{i(n-m)x}dx=0$$

if $n\neq m$ and $=1$ if $n=m$. □

Sturm-Liouville problems lead to something very similar. Let $q\in C([0,1],\mathbb{R})$ and let $\lambda\in\mathbb{C}$. We consider the boundary value problem

$$-u''+qu=\lambda u \qquad \text{in } (0,1) \qquad u(0)=u(1)=0 \tag{2.12}$$

We have proven in *Einführung in die PDG*:

Theorem 2.16. *Given $\lambda\in C$ the space of solutions to* (2.12) *is vector space of dimension* 0 *or* 1. *There exists a monotone sequence $\lambda_n\to\infty$ and a sequence of real valued functions $u_n\in C^2([0,1])$ which satisfy*

$$-u_n''+qu_n=\lambda_n u_n$$

$$\int u_n u_m dx = \delta_{nm}.$$

The functions u_n have $n-1$ zeroes in $(0,1)$. If $\lambda \neq \lambda_n$ for some j then (2.12) has only trivial solutions.

We repeat the proof of orthogonality:

$$\lambda_n \int u_n \overline{u_m} dx = \int (-u_n'' + q u_n) \overline{u_m} dx = \int u_n \overline{(-u_m'' + q u_m)} dx = \bar{\lambda}_m \int u_n \bar{u}_m dx$$

which implies either $\lambda_n = \lambda_m$ (since λ_m is real) or $\int u_n u_m dx = 0$.

Lemma 2.17 (Bessel inequality)**.** *Let x_j be an orthonormal system. Then*

$$0 \le \|x\|^2 - \sum_{n=1}^N |\langle x, x_n\rangle|^2 = \|x - \sum_{n=1}^N \langle x, x_n\rangle x_n\|^2$$

Proof. Let $M \subset H$ be the N dimensional subspace spanned by the elements x_j and let p be the projection to the closest point. Then by Lemma 2.10

$$\|x\|^2 = \|x - p(x)\|^2 + \|p(x)\|^2.$$

Moreover

$$p(x) = \sum_{j=1}^N \lambda_j x_j$$

and

$$\langle x, x_n\rangle = \langle p(x), x_n\rangle = \sum_{m=1}^N \lambda_m \langle x_m, x_n\rangle = \lambda_n$$

and

$$\|p(x)\|^2 = \sum_{n=1}^\infty |\lambda_n|^2.$$

□

Definition 2.18. *A subset A of a metrix space X is called dense if its closure is X. A metric space X is called separable, if there is a countable dense subset.*

Examples:

1. $\mathbb{N}$ is countable.
2. X and Y countable implies $X \times Y$ is countable. In particular $\mathbb{Q}^d$ is countable.
3. If X_j are countable sets then there union is countable.

4. Subsets of countable sets are countable.
5. $\mathbb{Q}$ is countable.
6. $\mathbb{Q}^N$ is countable.
7. $\mathbb{R}^N$ is separable since Q^N is countable and dense.
8. $l^2(\mathbb{N})$ is separable.

Definition 2.19. *A orthonormal set x_j of a Hilbert space is called orthonormal basis if*

$$\langle x, x_j \rangle = 0 \quad \textit{for all } j \in \mathbb{N}$$

implies $x = 0$.

Theorem 2.20. *The following properties are equivalent for a Hilbert space H which is not finite dimensional.*

- *The space H is separable.*
- *There exists an orthonormal basis and $\|x\|^2 = \sum |\langle x, x_j \rangle|^2$.*
- *The exists a surjective isometry $l^2 \to H$*

Proof. Suppose that H is separable. Let (y_n) be a dense sequence and X_N the span of the first N y_n. Its dimension is at most N. We use the Gram-Schmidt procedure to find an orthonormal basis (x_n) of X_N. We do this recursively by increasing n. This leads to a countable orthonormal sequence (x_n) so that its span is dense. Now let $x \in X$. By the Bessel inequality

$$\sum_{j=1}^N \langle x, x_j \rangle^2 + \|x - \sum_{j=1}^N \langle x, x_j \rangle x_j\|^2 = \|x\|^2.$$

Thus

$$N \to \|x - \sum_{n=1}^N \langle x, x_n \rangle x_n\|$$

is monotonically decreasing. Since (y_n) is dense and $\sum_{n=1}^N \langle x, x_n \rangle x_n = p_N x$ is the closed point in the span of $(x_n)_{n \le N}$ it converges to 0, which is equivalent to

$$\sum_{n=1}^N \langle x, x_n \rangle x_n \to x$$

in L^2, which in turn implies

$$\|x\|^2 = \sum_{n=1}^\infty |\langle x, x_n \rangle|^2.$$

Now suppose that (x_n) is an orthonormal basis. We want to define

$$l^2 \ni (a_n) \to \sum_{n=1}^{\infty} a_n x_n \in H.$$

We claim that for $M \geq N$

$$\sum_{n=1}^{N} a_n x_n - \sum_{n=1}^{M} a_n x_n = \sum_{n=N+1}^{M} a_n x_n$$

the norm of which is given by

$$\sqrt{\sum_{n=N+1}^{M} |a_j|^2}.$$

This implies that the partial sums are a Cauchy sequence and we define

$$x := \lim_{N\to\infty} \sum_{j=n}^{N} a_n x_n$$

Then

$$\|x\|^2 = \lim_{N\to\infty} \sum_{n=1}^{N} |a_n|^2 = \|(a_n)\|^2.$$

The map is clearly linear, surjective (by the previous part) and an isometry.

To complete the proof it suffices to prove that $l^2(\mathbb{N})$ is separable. Almost by definition the (countable) union of the subspaces of dimension N of sequence being 0 for indices $> N$ are dense: The truncated series converge in L^2. It suffices now to find a dense countable subset of $\mathbb{R}^N$. $\mathbb{Q}^N$ is an obvious choice.

□

In particular a Hilbert space is either isomorphic (there exists an isometric surjective linear map) to $\mathbb{R}^d$ resp. $\mathbb{C}^d$, to l^2, or it is not separable.

Example: The space $l^2(\mathbb{R})$ with inner product

$$\langle f, g \rangle = \sum_{x\in\mathbb{R}} f(x)\overline{g(x)}$$

is not separable since the vectors

$$e^x_y = \begin{cases} 0 & \text{if } y \neq x \\ 1 & \text{if } y = x \end{cases}$$

are an uncountable orthonormal system. In particular the pairwise distance is $\sqrt{2}$ and there cannot be a countable dense set.

There are natural questions:

- Which of the concrete orthonormal systems constructed above are a basis? We will see that the answer is all of them, but we need more tools to prove this.
- Is there a good theory of not necessarely orthonormal basis? This is more tricky.

[02.11.2016]

[04.11.2016]

3 Lebesgue spaces

3.1 Review of measure spaces

Reference:

1. Alt: Linear functional analysis, Springer.
2. Lieb and Loss: Analysis, AMS 2001.
3. Sharkarchi and Stein: Real Analysis: Measure theory, Integration and Hilbert spaces. Princeton University Press. 2009.

Theorem 3.1 (Banach-Tarski)**.** *There exists finitely many pairwise disjoints sets A_n, B_m of $\mathbb{R}^3$ and isometric maps $\phi_j, \psi_j \colon \mathbb{R}^3 \to \mathbb{R}^3$ so that*

$$B_1(0) = \bigcup_{n=1}^{N} \phi_n(B_n) = \bigcup_{m=1}^{M} \psi_m(A_m) = \bigcup_{n=1}^{N} B_n \cup \bigcup_{m=1}^{M} A_m$$

Remark: Makes use of the axiom of choice.

Definition 3.2. *Let X be a set. A family of subset $\mathcal{A}$ is called a σ algebra if*

1. $\{\} \in \mathcal{A}$
2. *$A \in \mathcal{A}$ implies $X \backslash A \in \mathcal{A}$*
3. *$A_n \in \mathcal{A}$ implies $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{A}$.*

A map $\mu : \mathcal{A} \to [0, \infty]$ is called a measure if whenever $A_n \in \mathcal{A}$ are pairwise disjoint then

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$

The triple $(X, \mathcal{A}, \mu)$ is called a measure space.

Examples:

1. X a set, $\mathcal{A} = 2^X$ the set of all subsets, and $\mu(A)$ the number of elements.

2. If (X, d) is a metric space then there is a smallest σ algebra containing all open sets. It is called the Borel σ algebra of X.

3. In probability theory the σ algebra encodes the available information on a system.

4. $X = \mathbb{R}^n$, $\mathcal{A}$ the Borel sets, μ the Lebesgue measure restricted to the Borel sets.

5. $X = \mathbb{R}^n$, $\mathcal{A}$ the Lebesgue sets, μ the Lebesgue measure.

6. $X = \mathbb{R}^n$, $0 \le s \le n$, $\mathcal{H}^s$, $\mathcal{A}$ the Borel sets, $\mathcal{H}^s$ the Hausdorf measure.

Definition 3.3. *Let X be a set and $\mathcal{A}$ a σ algebra. A map $f : X \to \mathbb{R} \cup \{-\infty, \infty\}$ is called measurable if*

$$f^{-1}((t, \infty]) \in \mathcal{A}$$

for all $t \in \mathbb{R}$. If $(X, \mathcal{A}, \mu)$ is a measure space and $f : X \to [0, \infty]$ then we define the Lebesgue integral by the Riemann integral

$$\int f d\mu = \int_0^\infty \mu(f^{-1}((t, \infty]))dt \in [0, \infty]$$

We call a measurable function f integrable if $|f|$ is integrable. Let $1 \le p < \infty$. We call a measurable function f p integrable if $|f|^p$ is integrable and denote

$$\|f\|_{L^p} = \left(\int |f|^p d\mu \right)^{1/p}.$$

We call a measurable function ∞ integrable or essentially bounded if there is a constant C so that

$$\mu(\{x : |f(x)| > C\}) = 0.$$

The best constant is denoted by $\|f\|_{L^\infty}$.

There are convergence theorems about the relation between the limit of integrals, and the integral over limits: The *theorem of Lebesgue* on dominated convergence, the *Lemma of Fatou* and the *theorem of Beppo Levi on monoton convergence.*

Definition 3.4. *A measure space $(X, \mathcal{A}, \mu)$ is called sigma finite if there exists a sequence of measurable sets A_j of finite measure so that $X = \bigcup\limits_{n=1}^{\infty} A_n$.*

3.2 Construction of measures

Measures are often constructed by first constructing *outer measures.*

Definition 3.5. *Let X be a set. An outer measure μ maps subsets of X to $[0, \infty]$ so that*

1. $\mu(\{\}) = 0$.
2. $A \subset B$ *implies* $\mu(A) \le \mu(B)$.
3. $\mu\Big(\bigcup\limits_{j=1}^{\infty} A_j\Big) \le \sum_{j=1}^{\infty} \mu(A_j)$.

Examples:

- Let $X = \mathbb{R}^d$. We define the measure of a coordinate rectangle as the product of the sidelengths and the measure of a countable disjoint union of coordinate rectangles as the sum over the measures of the rectangles. Finally we define the *outer* measure of a general set as the infimum of all measures of coverings by unions of coordinate rectancles.
- If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ finite measure spaces we define rectangles in the cartesian product as the cartesian product of measurable sets and their measure as the product of the measures. Then we proceed in the same way as above to obtain an outer measure on $X \times Y$.
- The Hausdorff measure: Let X be a metric space and $s \ge 0$. We define the premeasure of a set A of diameter r

$$\phi(A) = 2^{-s} \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)} r^s$$

and the Hausdorff measure

$$\mathcal{A} = \inf \Big\{ \sum_{n=1}^{\infty} \phi(A_n) : A \subset \bigcup_{n=1}^{\infty} A_n \Big\}.$$

Definition 3.6. *Let X be a metric space. We call μ an outer metric measure if it is an outer measure which satisfies*

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

for all $A, B \subset X$ with $\operatorname{dist}(A, B) > 0$.

Definition 3.7. *Let μ be an outer measure on X. We call a subset $A \subset X$ Caratheodory measurable if for all $B \subset X$*

$$\mu(B) = \mu(B \cap A) + \mu(B \cap (X \backslash A)).$$

Theorem 3.8 (Caratheodory)**.** *Let μ be an outer measure on the set X. Then the Caratheodory measurable sets $\mathcal{C}$ are a σ algebra and $(X, \mathcal{C}, \mu|_{\mathcal{C}})$ is a measure space. Moreover $\mathcal{C}$ contains all sets of exterior measure 0. If X is a metric space and μ is a metric outer measure than $\mathcal{C}$ contains all open sets. In the case of the Cartesian product $\mathcal{C}$ contains all Cartesian products of measurable sets.*

Theorem 3.9 (Fubini-Tonelli)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces, $\mathcal{A} \times \mathcal{B}$ the product σ algebra and $\mu \times \nu$ the product measure. Let f be $\mu \times \nu$ integrable. Then for almost of $x \in X$ $y \to f(x,y)$ is ν integrable, $x \to \int_Y f(x,y) d\nu(y)$ is μ integrable and*

$$\int_{X \times Y} f(x,y) d\mu \times \nu = \int_X \int_Y f(x,y) d\nu(y) d\mu(x).$$

[04.11.2016]

[09.11.2016]

3.3 Jensen's and Hölder's inequalities

Lemma 3.10. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is convex. Then both one sided derivatives exist and if $x < y$ then*

$$\frac{df}{dx}^+(x) \le \frac{df}{dy}^-(y) \le \frac{df}{dy}^+(y)$$

and for all z

$$f(z) \ge \max\{f(x) + \frac{df}{dx}^+(x)(z-x), f(x) + \frac{df}{dx}^-(x)(z-x)\}.$$

Proof. If $x_0 < x_1 < x_2$ then

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \le \frac{f(x_2) - f(x_0)}{x_2 - x_0} \le \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

and

$$h \to \frac{f(x+h) - f(x)}{h}$$

is monotonically increasing. This implies the differentiability from the right, and similarly from the left and the relation between the derivatives. □

Lemma 3.11. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $\mu(X) = 1$, $F : \mathbb{R} \to \mathbb{R}$ convex and f real valued and integrable. Then $F \circ f$ is measurable, $(F \circ f)_-$ is integrable and*

$$F \circ \int_X f d\mu \le \int_X F \circ f d\mu$$

Proof. Let $t_0 = \int_X f d\mu$. Since $F : \mathbb{R} \to \mathbb{R}$ is convex, we have for any t

$$F(t) \geq F(t_0) + \frac{dF^+}{dt}(t_0)(t - t_0).$$

Thus

$$\mu(\{F \circ f \leq s\}) \leq \mu(\{F(t_0) + \frac{dF^+}{dt}(t_0)(f - t_0) \leq s\})$$

and $\min\{F \circ f, 0\}$ is integrable since $x \to F(t_0) + \frac{dF^+}{dt}(t_0)(f - t_0)$ (which is affine in f) is integrable. Then

$$\begin{aligned}\int_X F \circ f d\mu \geq & \int_X F(t_0) + \frac{dF^+}{dt}(t_0)(f - t_0) d\mu \\ =& F(t_0) + \frac{dF^+}{dt}(t_0)\left(\int_X f d\mu - t_0\right) = F(t_0).\end{aligned}$$

□

We call a function $f : X \to \mathbb{C}$ integrable if the real and imaginary parts are both integrable. We say a property holds almost everywhere, if it holds outside a set of measure 0.

Lemma 3.12. *Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p$ and $g \in L^q$ then fg is integrable and*

$$\left| \int fg d\mu \right| \leq \|f\|_{L^p} \|g\|_{L^q}.$$

If $1 < p < \infty$, then the equality implies that $g = \lambda |f|^{p-2} \bar{f}$ almost everywhere for some $\lambda \in \mathbb{K}$ with $|\lambda| = 1$.

Proof. We copy the proof basically from the one for the sequence space. As there it suffices to consider f and g with $\|f\|_{L^p} = 1$ and $\|g\|_{L^q} = 1$ and prove

$$\int |f||g| d\mu \leq \int \frac{1}{p}|f|^p + \frac{1}{q}|g|^q d\mu = 1.$$

The inequality is strict unless

$$|fg(x)| = \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q$$

almost everywhere, which implies $|g| = |f|^{p-1}$. Now

$$\left| \int fg d\mu \right| \leq \int |fg| d\mu \leq \|f\|_{L^p} \|g\|_{L^q}$$

and in the case of equality all inequalities must be equalities. Hence $|g| = |f|^{p-1}$. Now suppose for some integrable function h

$$\left| \int h d\mu \right| = \int |h| d\mu.$$

Then there exits $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ so that $\int h d\mu \in [0, \infty)$ and

$$\int \lambda^{-1} h d\mu = \int \operatorname{Re} \lambda^{-1} h d\mu = \int |h| d\mu$$

and hence

$$h = \lambda |h|$$

almost everywhere. Back to our situation above this implies

$$g = \lambda |f|^{p-2} \bar{f}$$

almost everywhere for some complex number of modulus 1. □

3.4 Minkowski's inequality

Theorem 3.13. *Let $1 \le p < \infty$ and let X and Y be spaces with σ finite measures μ and ν respectively. Let f be $\mu \times \nu$ measurable. Then*

$$\left(\int_X \left| \int_Y |f(x,y)| d\nu(y) \right|^p d\mu(x) \right)^{1/p} \le \int_Y \left(\int_X |f(x,y)|^p d\mu(x) \right)^{1/p} d\nu(y).$$

If $1 < p$, if the integrals above are finite, and if

$$\left(\int_X \left| \int_Y f(x,y) d\nu(y) \right|^p d\mu(x) \right)^{1/p} = \int_Y \left(\int_X |f(x,y)|^p d\mu(x) \right)^{1/p} d\nu(y).$$

then there exist a μ-measurable function α and a ν-measurable function β so that

$$f(x,y) = \alpha(x)\beta(y)$$

almost everywhere. A special case is the triangle inequality (which holds without assuming σ finiteness)

$$\|f + g\|_{L^p(\mu)} \le \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)},$$

whenever f and g are p-integrable, with equality for $p > 1$ iff f and g are linearly dependent.

Proof. We assume first that f is nonnegative and omit the absolute value. We claim that

$$y \to \int_X f^p(x,y) d\mu(x) \qquad \text{and } H(x) := \int_Y f(x,y) d\nu(y)$$

are measurable functions. This follows from the Theorem of Fubini if f resp f^p are $\mu \times \nu$ integrable, and by an approximation argument in the general case. Then

$$\begin{aligned}\int_X H^p(x)d\mu(x) &= \int_X\int_Y f(x,y)d\nu(y)H(x)^{p-1}d\mu(x)\\ &= \int_Y\int_X f(x,y)H^{p-1}(x)d\mu(x)d\nu(y)\\ &\le \int_Y \Big(\int_X f^p(x,y)d\mu(x)\Big)^{1/p}\Big(\int H^p d\mu(x)\Big)^{\frac{p-1}{p}} d\nu(y)\\ &= \int_Y \Big(\int_X f^p(x,y)d\mu(x)\Big)^{1/p} d\nu(y)\Big(\int H^p d\mu(x)\Big)^{\frac{p-1}{p}}\end{aligned}$$

where we used Hölder's inequality with $q = \frac{p}{p-1}$.

We want to divide by the right hand. We can do that whenever the left hand side is neither 0 nor ∞, and we can achieve that in the same fashion as for sequences.

Now assume that $p > 1$, f is complex valued and integrable. Then, with

$$\int_X \Big|\int_Y f(x,y)d\nu(y)\Big|^p d\mu(x) \le \int_X \Big(\int_Y |f(x,y)|d\nu(y)\Big)^p d\mu(x)$$

and we continue as in the previous step, assuming equality. Then we have equality in the application of Hölder's inequality

$$|f(x,y)| = \alpha_0(x)\int_Y |f(x,y')|d\nu(y')$$

for almost all x and y. Since we must have also equality in the equality above we must have

$$f(x,y) = \alpha(x)\beta(y)$$

for some measurable function α and β.

For the last part we apply the first part with the counting measure on $Y = \{0,1\}$. The product measure is defined in the obvious fashion, even without assuming that μ is σ finite. If f is p integrable then by the definition of the integral

$$\mu(\{x : |f(x)| > t\}) \le t^{-p}\|f\|_{L^p}^p$$

Let

$$A = \bigcup_{j=1}^{\infty}\{x : |f(x)| + |g(x)| > \frac{1}{j}\}$$

which is a countable union of sets of finite measure. We replace X by A, take as σ algebra the sets in $\mathcal{A}$ which are contained in A, and μ restricted to this σ algebra as measure. This is σ additive. $\square$

3.5 Hanner's inquality

There is an improvement of the triangle inequality.

Theorem 3.14 (Hanner's inequality). *Let $(X, \mathcal{A}, \mu)$ be a measure space and f, g be p-integrable functions, $1 < p < \infty$. If $1 \le p \le 2$ then*

$$\|f+g\|_{L^p}^p + \|f-g\|_{L^p}^p \ge (\|f\|_{L^p} + \|g\|_{L^p})^p + \big|\|f\|_{L^p} - \|g\|_{L^p}\big|^p, \tag{3.1}$$

$$(\|f+g\|_{L^p} + \|f-g\|_{L^p})^p + \big|\|f+g\|_{L^p} - \|f-g\|_{L^p}\big|^p \le 2^p(\|f\|_{L^p}^p + \|g\|_{L^p}^p). \tag{3.2}$$

If $2 \le p < \infty$ all inequalities are reversed.

The inequalities reduce to the parallelogram identity if $p = 2$. Both are equivalent: The second is obtained from the first by replacing f by $f+g$ and g by $f-g$. It suffices to prove the first inequality.

Proof. We may assume that $\|g\|_{L^p} \le \|f\|_{L^p}$ (otherwise we exchange the two) and $\|f\|_{L^p} = 1$ (otherwise we multiply f and g by the inverse of the norm).

The first inequality follows from the following pointwise inequality: Let

$$\alpha(r) = (1+r)^{p-1} + (1-r)^{p-1}, \quad \beta(r) = [(1+r)^{p-1} - (1-r)^{p-1}]r^{1-p}.$$

We claim that

$$\alpha(r)|f|^p + \beta(r)|g|^p \le |f+g|^p + |f-g|^p \tag{3.3}$$

for $1 \le p \le 2$, $0 \le r \le 1$ and complex numbers f and g (and the reverse inequality for $2 \le p < \infty$). Indeed, (3.3) implies

$$\alpha(r)|f(x)|^p + \beta(r)|g(x)|^p \le |f(x)+g(x)|^p + |f(x)-g(x)|^p$$

and by integration

$$\alpha(r)\|f\|_{L^p}^p + \beta(r)\|g\|_{L^p}^p \le \|f+g\|_{L^p}^p + \|f-g\|_{L^p}^p.$$

We apply the inequality with $r = \|g|_{L^p}$ and recall that $\|f\|_{L^p} = 1$. The left hand side becomes

$$\begin{aligned}\Big[(\|f\|_{L^p} + \|g\|_{L^p})^{p-1} + (\|f\|_{L^p} - \|g\|_{L^p})^{p-1}\Big]\|f\|_{L^p} \\ + \Big[(\|f\|_{L^p} + \|g\|_{L^p})^{p-1} - (\|f\|_{L^p} - \|g\|_{L^p})^{p-1})\Big]\|g\|_{L^p} \\ =(\|f\|_{L^p} + \|g\|_{L^p})^p + (\|f\|_{L^p} - \|g\|_{L^p})^p.\end{aligned}$$

It remains to prove (3.3).

Let for $0 \le R \le 1$,

$$F_R(r) = \alpha(r) + \beta(r)R^p.$$

We claim that it attains its maximum at $r = R$ if $1 \le p < 2$ and resp. its minimum if $p > 2$. We compute

$$F_R' = \alpha' + \beta' R = (p-1)[(1+r)^{p-2} - (1-r)^{p-2}](1-(R/r)^p)$$

and the derivative vanishes only at $r = R$ and changes sign there. Thus

$$\alpha(r) + \beta(r) R^p \le (1+R)^p + (1-R)^p$$

if $0 \le R \le 1$ and $p \le 2$ with the opposite inequality if $p \ge 2$. Now let $R \ge 1$. Since $\beta \le \alpha$ if $p \le 2$ we obtain

$$\alpha(r)+\beta(r)R^p \le R^p\alpha(r)+\beta(r) \le R^p[(1+R^{-p})+(1-R^{-p})] = (1+R^p)+(R^p-1)$$

and the reverse inequality if $p > 2$. This implies (3.3) for real f and g. We claim that (3.3) holds for complex f and g. It suffices to consider $f = a > 0$ and $g = be^{i\theta}$. Since

$$(a^2 + b^2 + 2ab\cos\theta)^{p/2} + (a^2 + b^2 - 2ab\cos\theta)^{p/2}$$

has its minimum at $\theta = 0$ (resp. its maximum if $p \ge 2$) since $x \to x^{p/2}$ is concave if $p \le 2$ (convex if $p \ge 2$).

□

[09.11.2016]

[11.11.2016]

3.6 The Lebesgue spaces $L^p(\mu)$

Lemma 3.15. *The set of p-integrable functions is a vector space. The Minkowski inequality*

$$\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}$$

holds. Moreover

$$\|\lambda f\|_{L^p} = |\lambda| \|f\|_{L^p}$$

and

$$\|f\|_{L^p} = 0$$

if and only if f vanishes outside a set of zero

$$\mu(\{f \ne 0\}) = 0.$$

These functions are p integrable for all p. They are a subvector space.

Proof. The vector space property follows from the Minkowski inequality. The other statements are obvious. □

Definition 3.16. *We call two measurable functions equivalent $f \sim g$ if $\mu(\{f \neq g\}) = 0$. We define $L^p(\mu)$ as the space of equivalence classes of p integrable functions.*

If $f \sim g$ then $\|f-g\|_{L^p} = 0$. The equivalence relation is compatible with the vector space structure.

Theorem 3.17. *[Fischer-Riesz] The space $L^p(\mu)$ is Banach space.*

Proof. It is straight forward to verify that $L^p(\mu)$ is a vector space (using Minkowksi's inequality), and that $\|.\|_{L^p}$ is a norm. Completeness is more involved.

Let f_n be representatives of a Cauchy sequence. By taking subsequences if necessary we may assume

$$\|f_n - f_m\|_{L^p} \leq 2^{-\min\{m,n\}}.$$

We define the monotone sequence of functions

$$F_n(x) = |f_1(x)| + \sum_{m=1}^{n-1} |f_{m+1}(x) - f_m(x)|$$

and $F = \lim_{n\to\infty} F_n(x)$. F is measurable and by monotone convergence

$$\int |F|^p d\mu = \lim_{n\to\infty} \int |F_n|^p d\mu \leq \|f_1\|_{L^p}^p + 1$$

and in particular it is finite almost everywhere. Thus

$$f_n = f_1 + \sum_{m=1}^{n-1} (f_{m+1} - f_m)$$

converges if $F(x) < \infty$. Let f be the limit if $F(x) < \infty$, and 0 otherwise. It is measurable. Since $\max\{f, f_n\} \leq F$ we obtain by dominated convergence

$$\|f - f_n\|_{L^p}^p = \int |f - f_n|^p d\mu \to 0.$$

□

3.7 Projections and the dual of $L^p(\mu)$

Lemma 3.18. *Let $1 < p < \infty$ and let K be a closed convex set in $L^p(\mu)$. Let $f \in L^p(\mu)$. Then there exists a unqiue $g \in K$ with*

$$\|f - g\|_{L^p(\mu)} = \operatorname{dist}(f, K).$$

Moreover

$$\operatorname{Re} \int_X (h-g)(\bar f - \bar g)|f-g|^{p-2} d\mu \leq 0, \quad \forall h \in K.$$

Proof. Let h_n be a minimzing sequence. Since $\frac{1}{2}(h_n+h_m)\in K$ and $\|h_n-f+h_m-f\|_{L^p}\le \|h_n-f\|_{L^p}+\|h_m-f\|_{L^p}$, we see that

$$\|h_n-f+h_m-f\|_{L^p}\to 2\operatorname{dist}(f,K).$$

Now let $p\le 2$, from the second Hanner's inequality we obtain

$$\begin{aligned}\big(\|h_n-f+h_m-f\|_{L^p}&+\|h_n-h_m\|_{L^p}\big)^p\\ &+\big|\|h_m-f+h_m-f\|_{L^p}-\|h_n-h_m\|_{L^p}\big|^p\\ \le& 2^p\big(\|h_n-f\|_{L^p}^p+\|h_m-f\|_{L^p}^p\big).\end{aligned}$$

Let $A=\limsup_{n,m\to\infty}\|h_n-h_m\|_{L^p}$. This limsup is obtained along two subsequences $n,m\to\infty$. Let $D=\operatorname{dist}(f,K)$. Then

$$(2D+A)^p+|2D-A|^p\le 2^{p+1}D^p$$

which implies $A=0$ by the strict convexity of $A\to|2D+A|^p$.

If $p>2$ we argue similarly with the first inequality.

Now let $g\in K$ be the point of minimal distance and let $h\in K$. Let

$$N(t)=\int|f-(g+t(h-g))|^p d\mu.$$

Then $N(t)$ attains its minimum at $t=0$ on the interval $[0,1]$. We claim that its derivative at $t=0$ is

$$\frac{d}{dt}N|_{t=0}=p\operatorname{Re}\int|f(x)-g(x)|^{p-2}(f(x)-g(x))(\bar g(x)-\bar h(x))d\mu.$$

This implies the assertion.

To calculate the derivative we assume that $f,g\in L^p(\mu)$ and define

$$N(t)=\|f+tg\|_{L^p}^p.$$

Since almost everywhere

$$\frac{d}{dt}|_{t=0}|f+tg|^p=p|f|^{p-2}\operatorname{Re} f\bar g$$

and the pth power is convex

$$|f|^p-|f-g|^p\le\frac{1}{t}(|f+tg|^p-|f|^p)\le|f+g|^p-|f|^p,$$

the formula follows by dominated convergence. $\square$

Theorem 3.19. *Let $(X,\mathcal{A},\mu)$ be a measure space, $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$j:L^q\ni g\to(f\to\int fg d\mu)\in(L^p(\mu))^*$$

is a conjugate linear isometric isomorphism.

The proof is the same as for Hilbert spaces: By Hölder's inequality the map is well defined and

$$\|j(f)\|_{(L^p)^*} \le \|f\|_{L^q}.$$

Since

$$j(f)(|f|^{q-2}\bar{f}) = \int |f|^q d\mu$$

we conclude as for Hilbert spaces that

$$\|j(f)\|_{(L^p)^*} \ge \|f\|_{L^q}.$$

Surjectivity is proven exactly as for Hilbert spaces.

[11.11.2016]

[16.11.2016]

Corollary 3.20. *Suppose that μ is σ finite. Then*

$$L^\infty \ni g \to (f \to \int fg d\mu) \in (L^1(\mu))^*$$

is an isometric isomorphism.

The proof is an exercise on sheet 5.

3.8 Young's inequality and Schur's lemma

Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose that $1 \le p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. If $f \in L^p(\mu)$, $g \in L^q(\mu)$ and $h \in L^r(\mu)$, then fgh is integrable and

$$\left| \int fgh d\mu \right| \le \|f\|_{L^p(\mu)} \|g\|_{L^q(\mu)} \|h\|_{L^r(\mu)}.$$

This is a consequence of a multiple application of Hölder's inequality:

$$\left| \int fgh d\mu \right| \le \|f\|_{L^p} \|gh\|_{L^{\frac{p}{p-1}}}$$

and

$$\int |g|^{\frac{p}{p-1}} |h|^{\frac{p}{p-1}} d\mu \le \| |g|^{\frac{p}{p-1}} \|_{L^{\frac{q(p-1)}{p}}} \|h\|_{L^{\frac{r(p-1)}{p}}} = \|g\|_{L^q}^{\frac{p}{p-1}} \|h\|_{L^r}^{\frac{p}{p-1}}$$

since

$$\frac{p}{p-1} \left(\frac{1}{q} + \frac{1}{r} \right) = \frac{p}{p-1} (1 - \frac{1}{p}) = 1.$$

We denote $L^p(\mathbb{R}^d)$ (or even L^p) for $L^p(m^d)$ where m^d is the Lebesgue measure.

Lemma 3.21. *Suppose that* $1 \le p, q, r \le \infty$ *satisfy*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$$

and that $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ *and* $h \in L^r(\mathbb{R}^d)$. *Then*

$$\mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \to f(-x)g(x-y)h(y)$$

is integrable and

$$I(f, g, h) := \int_{\mathbb{R}^d \times \mathbb{R}^d} f(-x)g(x-y)h(y) dm^{2d}(x, y)$$

satisfies

$$|I(f, g, h)| \le \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}$$

and

$$I(f, g, h) = I(g, f, h) = I(f, h, g) = I(h, g, f).$$

Proof. We assume $1 < p, q, r < \infty$ since the limit cases are simpler, and follow by obvious modifications. Measurability is a consequence of the theorem of Fubini. It suffices to prove the statement for nonnegative functions since

$$\int fgh dm^{2d} \le \int |fgh| dm^{2d}$$

and the integrability of fgh follows from the integrability of $|fgh|$. We define p', q' and r' by $\frac{1}{p} + \frac{1}{p'} = 1$, i.e. $p' = \frac{p}{p-1}$ etc. Let

$$\begin{aligned}
\alpha(x, y) =& |f(-x)|^{p/r'} |g(x-y)|^{q/r'}, \\
\beta(x, y) =& |f(-x)|^{p/q'} |h(y)|^{r/q'}, \\
\gamma(x, y) =& |g(x-y)|^{q/p'} |h(y)|^{r/p'}.
\end{aligned}$$

Then $\frac{1}{p'} + \frac{1}{q'} + \frac{1}{r'} = 1$ and

$$\begin{aligned}
I =& \int \alpha(x, y)\beta(x, y)\gamma(x, y) dm^{2d} \\
\le& \|\alpha\|_{L^{r'}} \|\beta\|_{L^{q'}} \|\gamma\|_{L^{p'}} \\
=& \|f\|_{L^p}^{\frac{p}{r'}} \|g\|_{L^q}^{\frac{q}{r'}} \|f\|_{L^p}^{\frac{p}{q'}} \|h\|_{L^r}^{\frac{r}{q'}} \|g\|_{L^q}^{\frac{q}{p'}} \|h\|_{L^r}^{\frac{r}{p'}} \\
=& \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}.
\end{aligned}$$

The second last equality is a consequence of the theorem of Fubini. $\square$

Theorem 3.22 (Young's inequality)**.** *Suppose that* $1 \le p, q, r' \le \infty$ *and*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r'}.$$

If $f \in L^p$ *and* $g \in L^q$*, then for almost all* x

$$f(x-y)g(y)$$

is integrable and

$$f * g(x) := \begin{cases} \int f(x-y)g(y)dm^d(y) & \text{if integrable} \\ 0 & \text{otherwise} \end{cases}$$

defines a unique element in $L^{r'}(\mathbb{R}^d)$ *and*

$$\|f * g\|_{L^{r'}(\mathbb{R}^d)} \le \|f\|_{L^p}\|g\|_{L^q}.$$

Proof. We have $e^{-|x|^2} \in L^r$ for all $1 \le r \le \infty$. Then

$$e^{-|x|^2} f(x-y)g(y)$$

is m^{2d} integrable by Lemma 3.21. We apply Fubini to see that $\int f(x-y)g(y)dm^d(y)$ exists for almost all x. By Theorem 3.19 the estimate follows once we prove

$$\left| \int f * g h dm^d \right| \le \|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}$$

for

$$\frac{1}{r} + \frac{1}{r'} = 1$$

and all $h \in L^r$. Since then

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$$

and, by Fubini and Lemma 3.21

$$\begin{aligned} \left| \int f * g h(x) dx \right| \le & \int |f| * |g||h| dm^d \\ = & \int |f(x-y)||g(y)||h(x)| dm^d(x,y) \\ \le & \|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}. \end{aligned}$$

□

There is a particular case: if $q = 1$ and $p = r'$:

$$\|f * g\|_{L^p} \le \|f\|_{L^p}\|g\|_{L^1}.$$

Schur's lemma gives a criterium for an integral kernel to define a linear map from $L^p(\nu)$ to $L^p(\mu)$ for $1 \le p \le \infty$.

Theorem 3.23 (Schur's lemma)**.** *Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ be σ finite measure spaces and $k : X\times Y \to \mathbb{R}$ be $\mu\times\nu$ measurable. Suppose that $C_1, C_2 \in [0,\infty)$ and*

$$\sup_x \int |k(x,y)|d\nu(y) \le C_1, \qquad \sup_y \int |k(x,y)|d\mu(x) \le C_2.$$

If $1\le p\le\infty$ and $f\in L^p(\nu)$, then

$$\int k(x,y)f(y)d\nu(y)$$

exists for almost all x and

$$\Big\| \int k(x,y)f(y)d\nu(y)\Big\|_{L^p(\mu)} \le C_1^{1-\frac{1}{p}} C_2^{\frac{1}{p}} \|f\|_{L^p(\nu)}.$$

The map

$$L^p(\nu) \ni f \to Tf := \int k(x,y)f(y)d\nu(y) \in L^p(\mu)$$

is a continuous linear map which satisfies

$$\|T\|_{L^p(\nu)\to L^p(\nu)} \le C_1^{1-\frac{1}{p}} C_2^{\frac{1}{p}}.$$

Proof. Repeating the argument of Young's inequality we have to prove that

$$\int |g(x)||k(x,y)||f(y)|d\mu\times\nu \le C_1^{1-\frac{1}{p}} C_2^{\frac{1}{p}} \tag{3.4}$$

where

$$\frac{1}{p}+\frac{1}{q}=1$$

if $\|f\|_{L^p} = \|g\|_{L^q} = 1$. If $p=1$ or $q=\infty$ this is an immediate consequence of the theorem of Fubini. So we assume $1<p,q<\infty$. It suffices to prove (3.4) for nonnegative functions f, g and k where f and g are bounded and 0 outside a set of finite measure, since then (3.4) follows by an approximation and monotone convergence.

For $z\in\mathbb{C}$ with $0\le \operatorname{Re} z\le 1$, we define

$$f_z = \begin{cases} |f|^{pz-1}f & \text{if } f\neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g_z = \begin{cases} |g|^{\frac{p}{p-1}(1-z)-1}g & \text{if } g\neq 0 \\ 0 & \text{otherwise .} \end{cases}$$

Then for $\sigma\in\mathbb{R}$,

$$\|f_{i\sigma}\|_{L^\infty} = \|g_{1+i\sigma}\|_{L^\infty} = 1,$$

and

$$\|f_{1+i\sigma}\|_{L^1} = \|f\|_{L^p}^p = \|g_{i\sigma}\|_{L^1} = \|g\|_{L^q}^q = 1,$$

and hence

$$\int |g_{i\sigma}(x)||k(x,y)||f_{i\sigma}(y)|d\mu \times \nu \le C_1$$

$$\int |g_{1+i\sigma}(x)||k(x,y)||f_{1+i\sigma}(y)|d\mu \times \nu \le C_2.$$

Moreover

$$f_{\frac{1}{p}} = f, \qquad g_{\frac{1}{p}} = g.$$

Notice that f_z and g_z are bounded and zero outside a set of finite measure. By dominated convergence

$$z \to H(z) = \int g_z(x)k(x,y)f_z(y)d\mu \times \nu$$

is continuous in the strip $\mathcal{C} = \{z : 0 \le \operatorname{Re} z \le 1\}$, differentiable and satisfies the Cauchy-Riemann differential equations. The claim follows from the three lines inequality:.

Lemma 3.24 (Three lines inequality)**.** *Suppose that $u \in C(\mathcal{C})$ is bounded and holomorphic in the interior. Then*

$$\sup_{\mathcal{C}} |u| = \sup_{\partial \mathcal{C}} |u|$$

We apply the lemma to

$$u(z) = C_1^{z-1} C_2^{-z} H(z).$$

□

[16.11.2016]

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Proof. 1) Let $U \subset \mathbb{C}$ be a bounded open connected set and $u \in C(\overline{U}; \mathbb{C})$ be a holomorphic function in the interior. We claim that then

$$\sup_{x \in \overline{U}} |u(x)| = \sup_{x \in \partial U} |u(x)|.$$

We prove this by contradiction. Suppose that $|u|$ attains its maximum M at some interior point z_0 and suppose that this is larger than $\sup_{\partial U} |u(z)|$. Then

$$f(z) = \operatorname{Re} u(z)/u(z_0)$$

satisfies $0 \le f \le M$ and $f(z_0) = M$. Moreover f is harmonic. Let

$$f_\varepsilon(x+iy) = f(x+iy) + \varepsilon|x - \operatorname{Re} z_0|$$

where ε is so small that $f_\varepsilon(z) < M$ for $z \in \partial U$. Then f_ε has a maximum in an interior point z_1. At this point the Hessian is negative semidefinite by its trace $\Delta f_\varepsilon(z_1) = 4\varepsilon$. This is a contradiction.

2) Let u be as in the lemma amd let

$$u_\varepsilon(z) = e^{\varepsilon z^2} u(z).$$

Since $u_\varepsilon(z) \to 0$ as $|\operatorname{Im} z| \to \infty$

$$\sup_{z \in \mathcal{C}} |u_\varepsilon(z)| = \sup_{z \in \partial\mathcal{C}} |u_\varepsilon| \le e^\varepsilon \sup_{z \in \partial\mathcal{C}} |u(z)|.$$

Now we let ε tend to zero. □

3.9 Borel and Radon measures

Let (X, d) be a metric space. We recall that the Borel sets $\mathcal{B}(X)$ are the smallest σ algebra containing all open sets.

Definition 3.25. *Let (X, d) be a metric space. A Borel measure is a measure on the Borel sets. A Radon measure is a Borel measure, such that for every $x \in X$ there exists an open environment $U \ni x$ so that $\mu(U) < \infty$ and such that for every Borel set A*

$$\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ compact } \}$$

Definition 3.26. *We call a measure complete, if the σ algebra contains every subset of a set of measure zero.*

The theorem of Fubini in the form stated holds for $\mu \times \nu$ with the smallest σ algebra containing all cartesian product of measurable sets. The Lebesgue measure restricted to the Borel sets is not complete. We can easily complete σ algebras.

Lemma 3.27. *Let μ be a Radon measure. Then the measure of compact sets is finite and for $\varepsilon > 0$ and K compact there exists an open set $U \supset K$ of finite measure with $\mu(U) \le \mu(K) + \varepsilon$.*

Proof. Let K be compact. For every $x \in K$ exists an open set U_x containing x with $\mu(U_x) < \infty$. Since K is compact and

$$K \subset \bigcup U_x$$

there exists a finite subcovering

$$K \subset \bigcup_{j=1}^{N} U_{x_j} =: U$$

and

$$\mu(K) \le \mu(U) \le \sum_{j=1}^{N} \mu(U_{x_j}).$$

We define $U_j = U \cap \{x : d(x, K) < \frac{1}{j}\}$. By the theorem of Lebesgue

$$\mu(U_j) \to \mu(K).$$

□

Definition 3.28. *We call X locally compact if for every point x there is a neighborhood whose closure is compact. We call (X, d) σ compact if it is locally compact and if it is a countable union of compact sets.*

Let (X, d) be σ compact. Then there exists a sequence of compact set K_j so that K_j is contained in the interior $\mathring{K}_{j+1}$ of K_{j+1} and $X = \bigcup_{j=1}^{\infty} K_j$. This is proven on exercise sheet 6.

Lemma 3.29. *Let μ be a Borel measure on a σ compact space (X, d) and let B be a Borel set with $\mu(B) < \infty$ and $\varepsilon > 0$. Then there exists a closed set $C \subset B$ with $\mu(B \backslash C) < \varepsilon$. If μ is in addition Radon then there exists an open set U containing B with $\mu(U \backslash B) < \varepsilon$.*

Proof. For the first part we may assume $\mu(X \backslash B) = 0$ - otherwise we define $\nu(A) = \mu(A \cap B)$. we define

$$\mathcal{F} = \left\{ \begin{array}{ll} A \subset \mathbb{R}^d : & A \text{ is Borel and for every } \varepsilon > 0 \text{ there exists a closed set } C \\ & \text{with } \mu(A \backslash C) < \varepsilon. \end{array} \right\}$$

It contains all closed sets. We claim:

1. If $A_j \in \mathcal{F}$ then $\bigcap A_j \in \mathcal{F}$.
2. If $A_j \in \mathcal{F}$ then $\bigcup A_j \in \mathcal{F}$.
3. Since open sets are countable unions of closed sets every open set is in $\mathcal{F}$.

We define

$$\mathcal{G} = \{A : X \backslash A, A \in \mathcal{F}\}$$

Then $\mathcal{G}$ contains complements of elements and countable unions of elements of $\mathcal{G}$. Hence it is a σ algebra containing all open sets, and thus it is the Borel σ algebra. This implies the first claim.

Let now μ be Radon and K_j as above. Then $K_j \backslash B$ is Borel with $\mu(\mathring{K}_j \backslash B) < \infty$. Then there exists a closed set $C_j \subset \mathring{K}_j \backslash B$ with $\mu((\mathring{K}_j \backslash C_j) \backslash B) < \varepsilon 2^{-j}$. Let

$$U = \bigcup_{j=1}^{\infty} (\mathring{K}_j \backslash C_j).$$

It is open and

$$B = \bigcup_{j=1}^{\infty} (\mathring{K}_j \cap B) \subset \bigcup \mathring{K}_j \backslash C_j = U$$

Moreover

$$\mu(U \backslash B) = \mu(\bigcup (\mathring{K}_j \backslash C_j) \backslash B) < \varepsilon.$$

□

Lemma 3.30. *Let (X, d) be σ compact, and μ a Borel measure such that any compact set is of finite measure. Then μ is Radon and it is outer regular.*

Proof. Only inner regularity has to be proven since outer regularity follows then by Lemma 3.29. Let A be Borel with finite measure (why does this suffice?). By Lemma 3.29 there exists a closed set $C \subset A$ such that $\mu(A \backslash C) < \varepsilon$. Let K_j be compact subsets with $X = \bigcup K_j$ and K_j contained in the interior of K_{j+1}. Then

$$\mu(C \cap K_j) \to \mu(C)$$

and $C \cap K_j$ is compact. □

The most important example is the Lebesgue measure. A Radon measure on a compact metric space is finite. If (X, d) is a countable union of compact sets and μ is a Radon measure then μ is σ finite.

The counting measure on $\mathbb{R}$ is not a Radon measure.

Remark 3.31. *Continuous functions on compact metric spaces are integrable with respect to Radon measures.*

Lemma 3.32. *Let (X, d) be a σ compact metric space, μ a Radon measure on X and $1 \le p < \infty$. Then continuous functions with compact support are dense in $L^p(\mu)$.*

Proof. Let f be integrable. We decompose it into real and the imaginary part and it suffices to prove the assertion for real functions. Similar we decompose a real valued function into positive and negative part, and it suffices to approximate a nonnegative integrable function f.

Since

$$\int f d\mu = \int_0^{\infty} \mu(\{f > t\}) dt$$

given $\varepsilon > 0$ there exists $0 = t_0 < t_1 < \dots t_j < t_{j+1} < t_N < \infty$ so that (with $t_0 = 0$)

$$0 < \int_0^\infty \mu(\{f > t\})dt - \sum_{j=1}^N (t_j - t_{j-1})\mu(\{f > t_j\}) < \varepsilon$$

Let

$$A_j = \{x : f(t) > t_j\}.$$

Then

$$\|f - \sum (t_j - t_{j-1})\chi_{A_j}\|_{L^1} < \varepsilon$$

and it suffices to approximate a characteristic function of a measurable set A of finite measure by a continuous function. Let $\varepsilon > 0$. By inner and outer regularity there exists a compact set K and an open set U so that

$$K \subset A \subset U \qquad \mu(U) \leq \mu(K) + \varepsilon.$$

Then $d(K, X \backslash U) := d_0 > 0$ and we define

$$f_L(x) = \max\{1 - Ld(x, K), 0\} \in C(X)$$

Then if $d_0 L \geq 1$

$$\|f_L - \chi_A\|_{L^p} < \varepsilon^{\frac{1}{p}}.$$

If L is sufficiently large then $\operatorname{supp} f$ is compact. Thus continuous functions with compact support are dense. $\square$

[18.11.2016]

[23.11.2016]

3.10 Compact sets

Lemma 3.33. *If (X, d) is σ compact and μ is Radon measure, then Lipschitz continuous functions with compact support are dense in $L^p(\mu)$ for $1 \leq p < \infty$.*

Proof. We prove that for every $\varepsilon > 0$ and $f \in C(X)$ with compact support, there exists f_ε Lipschitz continuous with

$$\operatorname{supp} f_\varepsilon \subset \operatorname{supp} f$$

and

$$\sup |f_\varepsilon - f| < \varepsilon.$$

It suffices to do this for $f \geq 0$. Since $\operatorname{supp} f$ is compact it is uniformly continuous: There exists $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ if $d(x, y) < \delta$. With

$$L = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : d(x, y) \geq \delta \right\}$$

which is finite since it is the supremum of a continuous function on a compact set, we obtain the inequality

$$|f(x) - f(y)| \leq \varepsilon + Ld(x, y), \quad \forall x, y \in X.$$

We define

$$g(x) = \min_y \{f(y) + 2Ld(x, y)\}.$$

One easily checks that g has Lipschitz constant $2L$, and the mimimum is attained in $B_\delta(x)$ and

$$\max\{0, f(x) - \varepsilon\} \leq g(x) \leq f(x).$$

$\square$

Theorem 3.34 (Arzela-Ascoli). *Let (X, d) be a compact metric space. Then a closed set $A \subset C_b(X)$ is compact if and only if*

1. *A is bounded.*
2. *A is equicontinuous, i.e. for $\varepsilon > 0$ there exists $\delta > 0$ so that*

$$|f(x) - f(y)| < \varepsilon \qquad \text{if } f \in A \text{ and } d(x, y) < \delta.$$

Proof. Let A be compact. Since

$$C_b(X) \ni f \to \|f\|_{C_b(X)}$$

is continuous and hence attains its maximum in A, we deduce that A is bounded. Let $\varepsilon > 0$. For every f there exists $\delta_f > 0$ and an open neighborhood $U_f \subset C_b(X)$ so that

$$|g(x) - g(y)| < \varepsilon \qquad \text{if } g \in U_f \text{ and } d(x, y) < \delta_f.$$

Then $A \subset_{f \in A} U_f$, and since A is compact there is a finite subcovering, $A \subset_{j=1}^N U_{f_j}$. We define $\delta = \min \delta_{f_j}$.

Now assume that A is closed, bounded and equicontinuous. Let $f_j \in A$ be a sequence. Given $\varepsilon > 0$ we claim that there exists g such that $B_{3\varepsilon}(g) \subset C_b(X)$ contains infinitely many f_j. Let $\varepsilon > 0$ and $\delta > 0$ as in the second condition. Then there exist a finite number N of points x_k so that $B_{\delta/2}(x_k)$ cover X since X is compact. There exists a subsequence so that $f_{j_l}(x_k)$ converges for all x_k. In particular, after relabeling, there are infinitely many $\{f_{j_l}\}_{l \in \mathbb{N}}$ so that $|f_{j_l}(x_k) - f_{j_m}(x_k)| < \varepsilon$. Then

$$f_{j_l} \subset B_{3\varepsilon}(f_{j_1}).$$

$\square$

Lemma 3.35. *Let (X, d) be a compact set. Then it is separable.*

Proof. Given ε there exists a finite number of points x_n^ε, $1 \le n \le N(\varepsilon)$ so that the union of the balls $B_\varepsilon(x_n^\varepsilon)$ cover X. Take a sequence $\varepsilon = 2^{-j}$. This leads to a dense sequence. $\square$

Corollary 3.36. *Let (X, d) be compact. Then $C_b(X)$ is separable.*

Proof. By the proof of Lemma 3.33 the Lipschitz continuous functions are dense. The countable union of separable sets is separable and its closure is separable. Hence it suffices to prove that

$$K = \{f \in C_b(X) : \|f(x)\|_{C_b(X)} \le n, |f(x) - f(y)| \le nd(x, y)\}$$

is separable. This set is compact by Theorem 3.34 and hence separable. $\square$

Corollary 3.37. *Let (X, d) be σ compact and μ a σ finite Borel measure. If $1 \le p < \infty$ then $L^p(\mu)$ is separable.*

Proof. Since Lipschitz continuous functions with compact support are dense we argue as for $C_b(X)$. $\square$

Corollary 3.38. *Suppose that $1 \le p < \infty$, $f \in L^p(\mathbb{R}^d)$, $\varepsilon > 0$. Then there exist $\delta > 0$ and $R > 0$ so that for all $|h| < \delta$*

$$\|f(. + h) - f(.)\|_{L^p} < \varepsilon, \quad \|\chi_{\mathbb{R}^d \setminus B_R(0)} f\|_{L^p} < \varepsilon.$$

Proof. The second claim is a consequence of monotone convergence. For the first we approximate f by a Lipschitz continuous function g with compact support, $\|g - f\|_{L^p} < \varepsilon/4$ and estimate

$$\begin{aligned} \|f(. + h) - f\|_{L^p} \le& \|f(. + h) - g(. + h)\|_{L^p} + \|f - g\|_{L^p} + \|g(\cdot + h) - g\|_{L^p} \\ \le& \varepsilon/4 + \varepsilon/4 + |h| \|g\|_{Lip} \Big(m^d(\operatorname{supp} g)\Big)^{1/p} \\ \le& \varepsilon \end{aligned}$$

by choosing $|h| \le r$ for some small r. $\square$

We want to characterize compact subsets of L^p spaces.

Theorem 3.39. *Let $1 \le p < \infty$. A closed subset $C \subset L^p(\mathbb{R}^n)$ is compact iff*

1. *C is bounded.*
2. *For every $\varepsilon > 0$ there exists δ so that for all $|h| < \delta$ and all $f \in C$*

$$\|f(. + h) - f\|_{L^p(\mathbb{R}^d)} < \varepsilon$$

3. For every $\varepsilon > 0$ there exists R so that for all $f \in C$

$$\|\chi_{\mathbb{R}^d \setminus B_R(0)} f\|_{L^p} < \varepsilon.$$

Proof. Let C be compact. Since $f \to \|f\|_{L^p}$ is continuous it attains its maximum and hence C is bounded. Suppose there exists $\varepsilon > 0$ and $h_j \to 0$ and $f_j \in C$ so that

$$\|f_j(. + h_j) - f_j\|_{L^p} \geq \varepsilon.$$

Since C is compact we may assume that f_j is a Cauchy sequence with limit f. Then there exists $\delta > 0$ so that

$$\|f(. + h) - f\|_{L^p} < \varepsilon/2$$

for $|h| < \delta$. This contradicts the previous inequality. Similarly we deduce the third part.

Vice versa: Suppose that $C \subset L^p(\mathbb{R}^d)$ is closed, bounded, and satisfies the three claims. We choose a smooth function η supported in the unit ball with values between 0 and 1 and $\int \eta = 1$, define $\eta_r(x) = r^{-d}\eta(x/r)$ and we fix $\varepsilon > 0$. Then there exists δ so that by Minkowski's inequality and the second assumption

$$\|f_r - f\|_{L^p} \leq \sup_{|h| \leq r} \|f(. + h) - f\|_{L^p} < \varepsilon, \quad f_r = \eta_r * f,$$

for all $f \in C$ and $r \leq \delta$. Moreover f_r is Lipschitz continuous with Lipschitz constant depending on δ. By Theorem 3.34 the set $\overline{C_\delta}$ is compact, and we can cover it by a finite number of balls of radius $\varepsilon/2$. But then the balls with radius ε cover C. Thus C is precompact, and compact since it is closed. $\square$

[23.11.2016]

[25.11.2016]

3.11 The Riesz representation theorem for $C_b(X)$

Definition 3.40. *Let (X, d) be a σ compact metric space and $K_j \subset X$ compact with K_j in the interior of K_{j+1} and $X = \bigcup K_j$. We denote by $C_0(X) \subset C_b(X)$ the continuous functions f with limit 0 at ∞, i.e. for all $\varepsilon > 0$ there exists j so that f is at most of size ε outside K_j. We define $C_c(X)$ as the subspace of continuous functions with compact support.*

Definition 3.41. *Let (X, d) be a metric space. We call $L \in (C_b(X))^*$ nonnegative if*

$$L(f) \geq 0 \qquad \text{whenever } f \geq 0.$$

Theorem 3.42. *Let (X, d) be a sigma compact metric space and let $L : C_c(X) \to \mathbb{K}$ satisfy*

$$|L(f)| \le C_K \|f\|_{C_b(X)}$$

for $\operatorname{supp} f \subset K$ *compact. Then there exists a Radon measure μ and a measureable function $\sigma : X \to \{\pm 1\}$ so that*

$$L(f) = \int_X f \sigma d\mu.$$

Definition 3.43. *Let L be as above. We define the variation measure of L by*

$$\mu^*(U) = \sup\{L(f) : f \in C_c(X), \operatorname{supp} f \subset U, |f| \le 1\}$$

for open sets U and for general sets

$$\mu^*(A) = \inf\{\mu(U) : A \subset U, U \text{ open }\}.$$

Proof. We prove the theorem by several steps.

1. μ^* is a outer metric measure, which defines a Radon measure on the Borel sets.

2. For $f \in C_c(X)$ nonnegative we define

$$\lambda(f) = \sup\{|L(g)| : |g| \le f\}$$

   and prove for f nonnegative

$$\lambda(f) = \int f d\mu$$

3. As a consequence

$$|L(f)| \le \int |f| d\mu$$

   and we can extend L to $L^1(\mu)$, hence $L \in (L^1(\mu))^*$ and there exists $\sigma \in L^\infty(\mu)$ so that

$$L(f) = \int f \sigma d\mu$$

   for all $f \in C_c(X)$ with $\|\sigma\|_{L^\infty(\mu)} \le 1$.

4. We complete the proof by $|\sigma(x)| = 1$ for almost all x. Since we may change on a set of μ measure 0, we obtain $|\sigma| = 1$.

We observe that by multiplying g by a constant of size 1 we may always assume that $L(g) \in [0, \infty)$ in the definition of μ^* and λ.

Step 1: We claim that μ^* is an outer metric measure. To show that it is an outer measure, let U_j be open sets, $U = \cup U_j$ and we have to show that

$$\mu^*(U) \le \sum \mu^*(U_j).$$

Let $0 \le f \le 1$ with $\operatorname{supp} f \subset U$. We have to show that

$$L(f) \le \sum \mu^*(U_j).$$

Let $K = \operatorname{supp} f$ which is compact. Thus K is covered by finitely many $\cup_{j=1}^N U_j$ for some $N < \infty$. Moreover we may assume that the U_j's are contained in a fixed compact set, or even replacing X by this compact set, that X is compact. We claim that there exist g_j, $0 \le g_j \le 1$, $\operatorname{supp} g_j \subset U_j$ and $\sum g_j = 1$ on K. We define $f_j = g_j f$. Then

$$L(f) = \sum_j L(f_j) \le \sum_{j=1}^N \mu^*(U_j).$$

To see the existence of the g_j, take $U_0 = X \backslash K$. Then $X = \cup_{j=0}^N U_j$ and we take a subordinate partition of unity, i.e. functions $\eta_j \in C_b(X)$ with $0 < \eta_j$ and $\operatorname{supp} \eta_j \in U_j$ so that

$$1 = \sum_{j=0}^N \eta_j.$$

The functions $g_j = \eta_j$ for $1 \le j \le N$ have this property. More precisely, let $A_0 = X \backslash \bigcup_{j=1}^N U_j$. It is compact and satisfies $A_0 \subset U_0$. There is $\tilde\eta_0 \in C_b(X)$ supported in U_0, identically 1 on A_0. Let $A_1 = X \backslash (\{x : \tilde\eta_0(x) > \frac12\} \bigcup_{j=2}^N U_j) \subset U_1$ and we repeat the contruction. Recursively we obtain $\tilde\eta_j$ with

$$\rho = \sum_{j=0}^N \tilde\eta_j \ge \frac12$$

in X. We define

$$\eta_j = \frac{\tilde\eta_j}{\rho}.$$

Finally, if A, B are Borel sets with positive distance there exist disjoint open sets V and W containing A resp. B. Then

$$\mu^*(A \cup B) = \inf \mu^*(U) = \inf \mu^*(U \cap V) + \mu^*(U \cap W) = \mu^*(A) + \mu^*(B).$$

Let μ be the measure defined by the Caratheodory construction. Then

$$\mu(U) = \mu^*(U)$$

for open sets. By construction μ is bounded on compact sets and thus its restriction to Borel sets is a Radon measure.

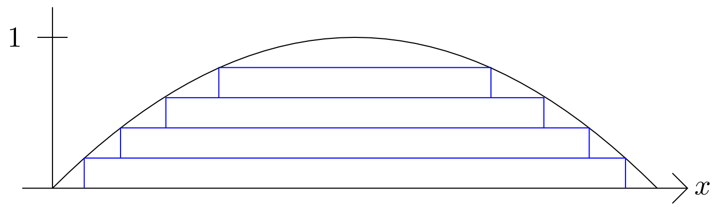


Figure 1: Formula (3.5)

Step 2: Let $f \in C_c(X)$ be non negative. We define

$$\lambda(f) = \sup\{|L(g)| : g \in C_c(X), |g| \le f\}.$$

Clearly $0 \le f_1 \le f_2$ implies $\lambda(f_1) \le \lambda(f_2)$ and for $c > 0$, $\lambda(cf) = c\lambda(f)$. We claim that

$$\lambda(f_1 + f_2) = \lambda(f_1) + \lambda(f_2)$$

for $f_1, f_2 \in C_c(X)$ nonnegative. Indeed, if $|g_1| \le f_1$ and $|g_2| \le f_2$ then $|g_1 + g_2| \le f_1 + f_2$, and, if in addition $L(g_1), L(g_2) \in [0, \infty)$,

$$|L(g_1) + L(g_2)| \le \lambda(f_1 + f_2).$$

This gives

$$\lambda(f_1) + \lambda(f_2) \le \lambda(f_1 + f_2).$$

Now let $|g| \le f_1 + f_2$. We define

$$g_1 = \begin{cases} \frac{f_1 g}{f_1 + f_2} & \text{if } f_1 + f_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

and similarly g_2. Then $|g_i| \le f_i$ and hence

$$|L(g)| \le \lambda(f_1) + \lambda(f_2)$$

which gives

$$\lambda(f) = \lambda(f_1) + \lambda(f_2).$$

We claim that

$$\lambda(f) = \int f d\mu.$$

It suffices to consider $0 \le f \le 1$. We approximate f by step function so that

$$\|f - \frac{1}{N} \sum_{j=1}^{N-1} \chi_{U_j} \|_{sup} < \frac{1}{N} \tag{3.5}$$

with

$$U_j = \{x : f(x) > \frac{j}{N}\}.$$

By continuity $U_{j+1} \subset U_j$. We approximate the characteristic function by continuous functions so that $\operatorname{supp} \eta_j \subset U_{j-1}$, $\eta_j = 1$ on U_j and $\mu(\operatorname{supp} \eta_j \backslash U_j) < 1/j$. It suffice to verify

$$\left| |\lambda(\eta_j)| - \int \eta_j d\mu \right| \leq 1/j$$

which follows from

$$\mu(U_j) \leq |\lambda(\eta_j)| \leq \mu(\operatorname{supp} \eta_j).$$

Step 3: Now

$$|L(f)| \leq \lambda(|f|) = \int |f| d\mu.$$

We extend L to an element in $(L^1(\mu))^*$, which is represented by an infinite integrable function σ by Corollary 3.20 . Moreover

$$\|\sigma\|_{L^\infty(\mu)} \leq \|L\|_{(L^1(\mu))^*} = 1.$$

Step 4: We claim that $|\sigma| = 1$ almost everywhere. By definition

$$\mu(U) = \sup\{\int f\sigma d\mu = L(f) : f \in C_c(X), |f| \leq 1, \operatorname{supp} f \subset U\}$$

We choose a sequence of functions with

$$\int f_j \sigma d\mu = L(f_j) \to \mu(U).$$

Since $\int f_j \sigma d\mu \leq \int |\sigma| d\mu$ and $|\sigma| \leq 1$ we deduce $|\sigma| = 1$ almost everywhere. $\square$

[25.11.2016]

[30.11.2016]

3.12 Covering lemmas and Radon measures on $\mathbb{R}^d$

The space $\mathbb{R}^d$ is σ compact and the Lebesgue measure is σ finite Radon measure.

Theorem 3.44 (Covering theorem of Besicovitch)**.** *There exists M_d depending only on d so that every family $\mathcal{F}$ of closed balls with bounded radii contains M_d subfamilies G_m, $1 \leq m \leq M_d$ so that each G_m consists disjoint balls and if A is the set of the centers then*

$$A \subset \bigcup_m \bigcup_{B \in G_m} B.$$

The same statement with the same proof holds for open balls.

Proof. We assume first that A is bounded and define D as the supremum of the radii. There exists a ball $B_1 = \overline{B_{r_1}(x_1)}$ with $r_1 \geq \frac{3D}{4}$. We choose recursively $B_n = \overline{B_{r_n}(x_n)}$ with x_n in

$$A_n = A \backslash \bigcup_{j=1}^{n-1} \overline{B_{r_j}(x_j)}$$

so that

$$r_n \geq \frac{3}{4} \sup\{r : \overline{B_r(x)} \in \mathcal{F}, x \in A_n\}.$$

We stop if $A_n = \{\ \}$. For simplicity we consider the case when the procedure does not stop. Then whenever $j \geq n$, we have $r_j \leq \frac{4}{3} r_n$ (otherwise we would not have chosen $\overline{B_{r_n}(x_n)}$) and

$$|x_j - x_n| \geq r_n \geq \frac{r_n + r_j}{3}$$

and the balls $B_{r_j/3}(x_j)$ are all disjoint and

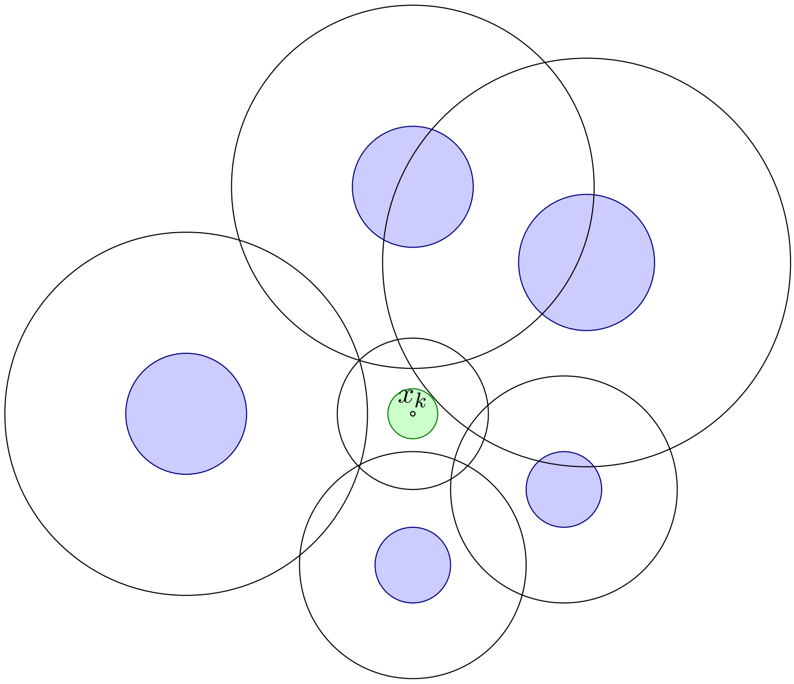
$$A \subset \bigcup B_j.$$

We fix $k > 1$ and define

$$I = \{j : 1 \leq j < k, B_j \cap B_k \neq \{\ \}\}.$$

We claim that there is a bound for the number of balls in I: $\#I \leq M_d$ with M_d depending only on d and D.

We first bound the number of small balls. Let $K := I \cap \{j : r_j \leq 3r_k\}$. Then $\#K \leq 20^d$.



To see that we consider $j \in K$ and choose $x \in B_{r_j/3}(x_j) \subset B_{5r_k}(x_k)$. The $\#K$ balls $B_{r_j/3}(x_j)$ are all disjoint and hence

$$(5r_k)^d \geq \sum_{j \in K} (r_j/3)^d \geq (r_k/4)^d \#K.$$

Next we bound the number of large balls, i.e $\#(I \backslash K)$. Let now $i, j \in I \backslash K$, $i \neq j$. We will give a *upper bound* on

$$\cos(\angle(x_k x_i, x_k x_j)) = \frac{\langle x_i - x_k, x_j - x_k \rangle}{|x_i - x_k||x_j - x_k|}.$$

This gives a lower bound on the distance of the points $\frac{x_n - x_k}{|x_n - x_k|}$ for $n < k$, $n \in I \backslash K$, and hence a upper bound on their numbers L_d depending only on the dimension since the unit sphere is compact. Therefore we can take $M_d = 20^d + L_d + 1$.

To simplify the notation we assume that $x_k = 0$. Let θ be the angle between the centers $\angle x_i, x_j$. Since $B_i \cap B_k \neq \{\ \}$ and $B_j \cap B_k \neq \{\ \}$, we have without loss of generality

$$|x_i| \leq |x_j|, |x_i| \leq r_i + r_k, |x_j| \leq r_j + r_k.$$

We claim that $x_i \in B_j$ if $\cos\theta > \frac{5}{6}$. Firstly we notice that if $|x_i - x_j| \geq |x_j|$, then

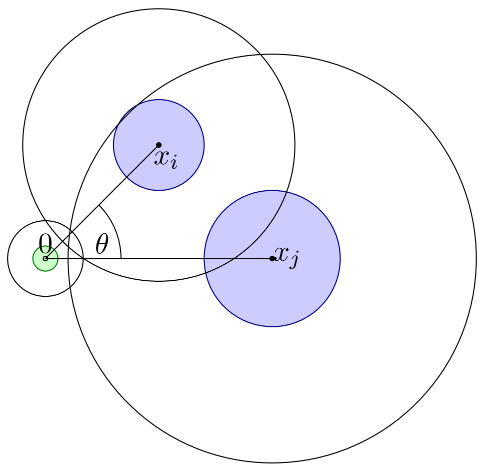
$$\cos\theta = \frac{|x_i|^2 + |x_j|^2 - |x_i - x_j|^2}{2|x_i||x_j|} \leq \frac{|x_i|^2}{2|x_i||x_j|} = \frac{|x_i|}{2|x_j|} \leq \frac{1}{2} \leq \frac{5}{6}.$$

Hence if we assume $\cos\theta \geq \frac{5}{6}$ then $|x_i - x_j| \leq |x_j|$. We suppose by contradiction that $x_i \notin B_j$. Then $r_j \leq |x_i - x_j|$ and

$$\begin{aligned}
\cos\theta =& \frac{|x_i|^2 + |x_j|^2 - |x_i - x_j|^2}{2|x_i||x_j|} \\
=& \frac{|x_i|}{2|x_j|} + \frac{(|x_j| - |x_i - x_j|)(|x_j| + |x_i - x_j|)}{2|x_i||x_j|} \\
\leq& \frac{1}{2} + \frac{|x_j| - |x_i - x_j|}{|x_i|} \\
\leq& \frac{1}{2} + \frac{r_j + r_k - r_j}{r_i} \\
=& \frac{1}{2} + \frac{r_k}{r_i} \leq \frac{5}{6}.
\end{aligned}$$

Now it suffices to derive the upper bound for $\cos\theta$ when $x_i \in B_j$, since otherwise $\cos\theta \leq \frac{5}{6}$ has already a upper bound. So we assume $x_i \in B_j$ from now on and we may restrict to $r_k = 1$ by scaling. Then $i < j$, since otherwise B_i would not have been chosen, and thus $x_j \notin B_i$, and

$$3 \leq r_i < |x_i - x_j| < r_j \leq \frac{4}{3} r_i, \quad r_i < |x_i| \leq 1 + r_i, \quad r_j < |x_j| < 1 + r_j.$$



The proof becomes now an exercise in planar geometry. We have

$$\begin{aligned}
\frac{3}{16} \le & \frac{1}{4}\frac{r_j}{|x_j|} \le \frac{1}{3}\frac{r_i}{|x_j|} \le \frac{\frac{2}{3}r_i - 1}{|x_j|} \le \frac{r_i + r_i - r_j - 1}{|x_j|} \\
\le & \frac{|x_i - x_j| + |x_i| - |x_j|}{|x_j|} \\
\le & \frac{|x_i - x_j| + |x_i| - |x_j|}{|x_j|} \frac{|x_i - x_j| - |x_i| + |x_j|}{|x_i - x_j|} \\
= & \frac{|x_i - x_j|^2 - ||x_i| - |x_j||^2}{|x_j||x_i - x_j|} \\
= & 2(1 - \cos\theta)\frac{|x_i||x_j|}{|x_j||x_i - x_j|} \\
= & 2(1 - \cos\theta)\frac{|x_i|}{|x_i - x_j|} \\
\le & 2(1 - \cos\theta)\frac{r_i + 1}{r_i} \\
\le & \frac{8}{3}(1 - \cos\theta)
\end{aligned}$$

and hence $\cos\theta \le \frac{119}{128}$.

It remain to define the sets G_m. We do this by defining a map

$$\sigma : \mathbb{N} \to \{1, \dots M_d\}.$$

We choose it to be the identity for $j \le M_d$. After that we proceed recursively, which we can do since

$$\#\Big\{ j \le k : B_j \cap B_{k+1} \ne \{\ \} \Big\} < M_d.$$

It remains to extend the result to unbounded sets. We do this by applying the first part in the annuli $6(m-1)D \le |x| < 6mD$.

□

Theorem 3.45. *Let μ be a Radon measure on $\mathbb{R}^d$ and let $\mathcal{F}$ be a family of closed balls and let A be a Borel set which is the union of the centers. We assume $\mu(A) < \infty$ and $\inf\{r : \overline{B_r(x)} \in \mathcal{F}\} = 0$ for $x \in A$. Let $U \subset \mathbb{R}^d$ be open. Then there exists a countable collection of disjoint closed balls $G \subset \mathcal{F}$ so that*

$$\mu\Big((A \cap U) \backslash \bigcup_{B \in G} B\Big) = 0.$$

Proof. We fix θ so that $1 - \frac{1}{M_d} < \theta < 1$ and claim that there is a finite collection of disjoint balls B_j, $1 \le j \le M$ in $\mathcal{F}$ so that

$$\mu\left((A \cap U) \backslash \bigcup_{j=1}^{M} B_j\right) \le \theta\mu(A \cap U).$$

Suppose this is true. Then we define

$$A_1 = A \backslash \bigcup_{j=1}^{M} B_j$$

and repeat the argument with $\mathcal{F}_1$ the subset of balls with center in A_1. After the kth step the complement has a measure at most $\theta^j \mu(U \cap A)$. So it remains to prove the claim. Let $\mathcal{F}_0$ be the subset of balls with radii at most 1. Then we apply the Besicovitch covering theorem and obtain G_m. Then

$$A \cap U \subset \bigcup_{j=1}^{M_d} \bigcup_{B \subset G_j} B$$

and

$$\mu(A \cap U) \leq \sum_{j=1}^{M_d} \mu\Big(A \cap U \cap \bigcup_{B \in G_j} B\Big).$$

There exists J so that

$$\frac{1}{M_d} \mu(A \cap U) \leq \mu\Big(A \cap U \cap \bigcup_{B \in G_J} B\Big).$$

By monotone convergence there exist finitely many balls in G_J so that the claim holds. $\square$

We turn to derivatives of Radon measures.

Definition 3.46. *Let μ and ν be Radon measures on $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ we define*

$$\overline{D_\mu} \nu(x) = \begin{cases} \limsup_{r \to 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) > 0 \text{ for all } r > 0 \\ \infty & \text{if for some } r > 0, \mu(B_r(x)) = 0 \end{cases}$$

$$\underline{D_\mu} \nu(x) = \begin{cases} \liminf_{r \to 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) > 0 \text{ for all } r > 0 \\ \infty & \text{if for some } r > 0, \mu(B_r(x)) = 0. \end{cases}$$

We say that ν is differentiable with respect to μ and x if $\overline{D_\mu} \nu(x) = \underline{D_\mu} \nu(x)$. Then we write $D_\mu \nu(x)$ and call this quantity the density of ν with respect to μ.

Remark 3.47. *Let $f \in C_c(\mathbb{R}^d)$ and μ Radon measure. Then*

$$x \to \int f(y - x) d\mu(y)$$

is continuous and hence Borel measurable.

Since the characteristic function of open and closed balls can be obtained as pointwise limit of continuous functions with compact support, the map

$$x \to \nu(B_r(x)), \quad x \to \mu(B_r(x))$$

are measurable. Thus

$$x \to \begin{cases} \frac{\nu(B_r(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) > 0 \\ \infty & \text{if } \mu(B_r(x)) = 0 \end{cases}$$

is Borel measurable. The map

$$r \to \begin{cases} \frac{\nu(B_r(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) > 0 \\ \infty & \text{if } \mu(B_r(x)) = 0 \end{cases}$$

is continuous from the left and right if $\mu(B_r(x)) > 0$ by inner and outer regularity. Thus also $\overline{D_\mu}\nu(x)$ and $\underline{D_\mu}\nu(x)$ are Borel measurable since we can write them as inf's and sup's over rational radii. Moreover by inner and outer regularity we obtain the same $\overline{D_\mu}\nu(x)$ and $\underline{D_\mu}\nu(x)$ if we use closed balls.

Theorem 3.48. *Let μ and ν be Radon measures on $\mathbb{R}^d$. Then*

1. *$D_\mu\nu(x)$ exists and is finite μ almost everywhere.*
2. *$D_\mu\nu$ is Borel measurable.*

[30.11.2016]

[02.12.2016]

Proof. We may assume that $\mu(\mathbb{R}^d) < \infty$ and $\nu(\mathbb{R}^d) < \infty$.

Step 1: We claim that for all Borel sets B and all $t > 0$

$$\nu(B \cap \{x : \underline{D_\mu}\nu(x) < t\}) \le t\mu(B \cap \{x : \underline{D_\mu}\nu(x) < t\})$$

and

$$\nu(B \cap \{x : \underline{D_\mu}\nu(x) > t\}) \ge t\mu(B \cap \{x : \underline{D_\mu}\nu(x) > t\}).$$

By outer regularity ($\mu(B) = \inf\{\mu(U) : B \subset U\}$) and it suffices to prove the assertion for $B = U$ open. Let $A = \{x \in U : \underline{D_\mu}\nu(x) < t\}$. Let

$$\mathcal{F} = \{\overline{B_r(a)} : a \in A, \overline{B_r(a)} \subset U, \nu(\overline{B_r(a)}) < t\mu(\overline{B_r(a)})\}.$$

For every $x \in A$, $\mathcal{F}$ contains arbitrarily small balls and we apply Theorem 3.45 to obtain a sequence of disjoint closed balls B_j in $\mathcal{F}$ so that

$$\nu(A \backslash \bigcup B_j) = 0$$

Then

$$\nu(A) = \sum \nu(B_j) \leq t \sum \mu(B_j) \leq t\mu(U).$$

Since

$$\mu(A) = \inf\{\mu(U) : A \subset U\}$$

we obtain the first inequality. The second one is proven similarly.

Step 2: We claim that $\overline{D_\mu}\nu(x) < \infty$ outside a set of μ measure 0. Let $A = \{x : \overline{D_\mu}\nu(x) = \infty\}$. Then

$$\nu(A) \geq t\mu(A)$$

for all t, hence $\mu(A) = 0$.

Step 3: For $s < t$ we define

$$R(s,t) = \{x : \underline{D_\mu}\nu(x) < s < t < \overline{D_\mu}\nu(x)\}$$

Then

$$t\mu(R(s,t)) \leq \nu(R(s,t)) \leq s\mu(R(s,t))$$

which implies $\mu(R(s,t)) = 0$. Since

$$\{x : \underline{D_\mu}\nu(x) < \overline{D_\mu}\nu(x)\} = \bigcup_{s<t, s,t \in \mathbb{Q}} R(s,t)$$

we see that $\underline{D_\mu}\nu(x) = \overline{D_\mu}\nu(x)$ for μ almost all x. □

Definition 3.49. *Let μ and ν be Borel measures on $\mathbb{R}^d$. We say the measure ν is absolutely continuous with respect to μ, $\nu << \mu$, if $\mu(A) = 0$ implies $\nu(A) = 0$. The mearures ν and μ are mutally singular with respect to μ if there exists a Borel set B such that $\mu(X \backslash B) = \nu(B) = 0$. We write then $\nu \perp \mu$.*

Theorem 3.50 (Radon-Nikodym)**.** *Let ν and μ be Radon measures on $\mathbb{R}^d$ with $\nu << \mu$. Then*

$$\nu(A) = \int_A D_\mu \nu d\mu$$

for all Borel sets.

Proof. It suffices to consider the case $\mu(\mathbb{R}^d) < \infty$ and $\nu(\mathbb{R}^d) < \infty$. We have seen that

$$\mu(\{D_\mu\nu(x) = \infty\}) = 0$$

and hence, since $\nu << \mu$, $\nu((\{D_\mu\nu(x) = \infty\}) = 0$. In the same fashion

$$\nu(\{D_\mu\nu(x) = 0\}) = 0.$$

Let A be a Borel set. For $t > 1$ we define

$$A_n = A \cap \{t^n \le D_\mu \nu < t^{n+1}\}.$$

Then

$$\nu(A) = \sum_{m=-\infty}^{\infty} \nu(A_m) \le \sum_{m=-\infty}^{\infty} t^{m+1}\mu(A_m) \le t\int_0^\infty \mu(\{D_\mu \nu > s\})ds = t\int_A D_\mu \nu d\mu$$

and

$$\nu(A) = \sum_{m=-\infty}^{\infty} \nu(A_m) \ge \sum_{m=-\infty}^{\infty} t^{m}\mu(A_m) \ge t^{-1}\int_0^\infty \mu(\{D_\mu \nu > s\})ds = t^{-1}\int_A D_\mu \nu d\mu.$$

We let now $t \to 1$. □

Theorem 3.51 (Lebesgue points)**.** *Let μ be a Radon measure on $\mathbb{R}^d$ and $\tilde f \in L^1_{loc}(\mu)$. Then*

$$f(x) := \lim_{r\to 0} \mu(B_r(x))^{-1} \int_{B_r(x)} \tilde f d\mu$$

exists almost everywhere and we define $f(x) = 0$ if it does not exist. Then f is in the equivalence class of $\tilde f$. If $\tilde f \in L^p_{loc}(\mu)$ then $f \in L^p_{loc}$ and

$$\lim_{r\to 0} \mu(B_r(x))^{-1} \int_{B_r(x)} |f(y) - f(x)|^p d\mu(y) = 0$$

almost everywhere.

Proof. It suffices to consider nonnegative $\tilde f$ and $\mu(\mathbb{R}^d) < \infty$. We define

$$\nu(A) = \int_A \tilde f \, d\mu.$$

This is a Radon measure by Lemma 3.30 which is absolutely continuous with respect to μ. Thus

$$\nu(A) = \int D_\mu \nu d\mu = \int \tilde f d\mu$$

and $D_\mu \nu$ lies in the equivalence class. Now the first claim follows from Theorem 3.48.

For every t, $|f(x) - t|^p$ is integrable. From the first part

$$\lim_{r\to 0} \mu(B_r(x))^{-1} \int_{B_r(x)} |f(y) - t|^p d\mu(y) = |f(x) - t|^p$$

almost every where. There is even a set N of μ measure zero so that this is true for all $t \in \mathbb{Q}$ outside the same set of measure zero. Let $\varepsilon > 0$. Thus the set of all x such that

$$\limsup_{r\to 0} \mu(B_r(x))^{-1} \int_{B_r(x)} |f(y) - f(x)|^p d\mu(y) > \varepsilon$$

is contained in N. To see this, chose $t \in \mathbb{Q}$ so that $|f(x) - t|^p < \varepsilon$. This completes the proof. □

Corollary 3.52. *Let μ be a Radon measure and $f \in L^p(\mu)$. Then there exists a cannonical representative of the equivalence class.*

4 Distributions and Sobolev spaces

4.1 Baire category theorem and consequences

Lemma 4.1 (Baire category theorem)**.** *A countable intersection of dense open subsets of a complete metric space is dense.*

Proof. Let (X, d) be a complete metric space and A_j open dense sets. Let $x \in X$ and $\varepsilon > 0$. Let $x_1 \in A_1$ so that $d(x, x_1) < \varepsilon/3$ and $0 < \delta_1 < \varepsilon/3$ so that $B_{2\delta_1}(x_1) \subset A_1$. We pick recursively x_n, δ_n so that $d(x_{n-1}, x_n) < \delta_n$, $2\delta_n < \varepsilon/3^n$ and $B_{2\delta_n}(x_n) \in A_n \cap B_{\delta_{n-1}}(x_{n-1})$.

By construction, $d(x_{n-1}, x_n) < \varepsilon/(2 \cdot 3^{n-1})$ and, if $n < m$, $d(x_n, x_m) < \frac{\varepsilon}{2\cdot 3^n} \sum_{j=0}^{m-n} 3^{-j} \leq \frac{3\varepsilon}{4\cdot 3^n}$ and (x_n) is a Cauchy sequence with limit y. Since $x_m \in \overline{B_{\delta_n}(x_n)}$ for $m \geq n$ the same is true for y, and $y \in A_n$ for all n. □

Theorem 4.2 (Banach-Steinhaus)**.** *Let X and Y be Banach spaces, $\mathcal{F} \subset L(X, Y)$. Suppose for each $x \in X$*

$$\sup\{\|Tx\|_Y : T \in \mathcal{F}\} < \infty.$$

Then

$$\sup\{\|T\|_{X\to Y} : T \in \mathcal{F}\} < \infty.$$

Proof. Let

$$C_n = \{x \in X : \sup_{T\in\mathcal{F}} \|Tx\|_Y \leq n\}.$$

This set is closed since both map and norm are continuous, and C_n is an intersection of closed sets. By assumption $\bigcup C_n = X$. We claim that some C_n has nonempty open interior. If not then the sets $U_n = X\backslash C_n$ are open and dense, with nonempty intersection, a contradiction to $\bigcup C_n = X$. Let $U \subset C_{n_0}$ be nonempty and open. It contains a ball $B_r(x_0)$. If $\|x\| < r$ then

$$\|Tx\|_Y \leq \|T(x - x_0)\|_Y + \|T(x_0)\|_Y \leq n_0 + \sup_{T\in\mathcal{F}} \|T(x_0)\|_Y =: R.$$

Then

$$\|T\|_{X\to Y} \le R/r$$

for all $T \in \mathcal{F}$. $\square$

[02.12.2016]

[09.12.2016]

The Baire category theorem has interesting further consequences.

Theorem 4.3. *Let X and Y be Banach spaces and $T \in L(X,Y)$. T is surjective if and only if it is open, i.e. if the image of open sets is open.*

Proof. Let T be open. Then $T(B_1(0))$ is open. In particular it contains a ball $B_r(0)$. Then $Y = \bigcup T(B_n(0))$ and T is surjective. Now suppose that T is surjective. It suffices to show that $T(B_1(0))$ contains a ball around 0. (Why?). Let

$$Y_n = T(B_n(0)) = \{Tx | \|x\|_X \le n\}.$$

It is closed by continuity, and $Y = \bigcup Y_n$. As above we conclude that one (and hence all) of the Y_n contains an open ball. Hence there exists $B_r(0) \subset Y_1$. $\square$

Corollary 4.4. *Suppose that $T \in L(X,Y)$ is injective and surjective. Then $T^{-1} \in L(Y,X)$.*

Thus continuous linear maps which are invertible as maps between sets are invertible as continuous linear maps.

Proof. Linearity of the inverse map is immediate. By Theorem 4.3 T is open. So $T(B_1^X(0))$ contains a ball $B_r^Y(0)$ and hence

$$\|Tx\|_Y \ge r^{-1}\|x\|_X.$$

$\square$

Theorem 4.5. *The set of nowhere differentiable functions in $C_b(0,1)$ is dense.*

Proof. Exercise. $\square$

4.2 Distributions: Definition

We need a preliminary result.

Lemma 4.6. *Let $U \subset \mathbb{R}^d$ be open and $k \in \mathbb{N}$. Then for every $f \in C_c^k(U)$ there exists a compact set $K \subset U$ and a sequence $f_n \in C^\infty(U)$ supported in K so that $\partial^\alpha f_n \to \partial^\alpha f$ in $C_b(U)$ for $n \to \infty$ and $|\alpha| \le k$.*

It suffices to consider $U = \mathbb{R}^d$. We will later prove a more general result and make sure that the reasoning is not circular.

Definition 4.7. *Let $U \subset \mathbb{R}^d$ be open, and $\mathcal{D}(U) = C_c^\infty(U)$ be the vector space of infinitely differentiable functions with compact support called test functions. We say $f_j \to f$ in $C_c^\infty(U) = \mathcal{D}(U)$ if there is a compact set $K \subset U$ and $\operatorname{supp} f_j \subset K$ for all j and for all multiindices α*

$$\partial^\alpha f_j \to \partial^\alpha f \qquad \text{in } C_b(U).$$

A distribution T on U is a continuous linear map from $C_0^\infty(U) \to \mathbb{K}$. We denote the space of distributions by $\mathcal{D}'(U)$.

By continuous we mean that

$$Tf_j \to Tf$$

if $f_j \to f$ in the sense of test functions. It is immediate that the distributions define a $\mathbb{K}$ vector space.

Lemma 4.8. *Let $T \in \mathcal{D}'(U)$. For every compact set $K \subset U$ there exists k and $C > 0$ so that, if $f \in \mathcal{D}(U)$ with $\operatorname{supp} f \subset K$ then*

$$|T(f)| \le C\|f\|_{C_b^k(U)}.$$

Proof. We define for $K \subset U$ compact

$$X_K = \{f \in \mathcal{D}(U) : \operatorname{supp} f \subset K\}.$$

We define a metric on X_K

$$d(f,g) = \sup_{k \ge 0} 2^{-k} \min\{1, \|f-g\|_{C_b^k(U)}\}.$$

With this metric X_K is a complete metric space:

$$d(f_j, f) \to 0 \qquad \text{iff } f_j \to f \qquad \text{in } C^k(U) \qquad \text{for all } k \ge 0.$$

Then $T \in \mathcal{D}'(U)$ define a continuous linear map from $X_K \to \mathbb{K}$. Moreover

$$|Tf| < \infty$$

for all $f \in X_K$. Now we argue as for the uniform boundedness principle of Banach-Steinhaus: There exists m so that the set

$$\{f \in X_K : |Tf| < m\}$$

contains an open ball. Then there exists $r > 0$ so that

$$|Tf| < m \qquad \text{for all } f \in X_K \quad \text{with } d(f,0) < r.$$

Let k be so that $2^{1-k} < r$. Then

$$d(f,0) < r \quad \text{if } \|f\|_{C^k(U)} < 2^{-k} \qquad \text{and } \operatorname{supp} f \subset K.$$

we continue as for the theorem of Banach-Steinhaus.

□

Definition 4.9. *We say that*

$$T_n \to T \qquad \text{in } \mathcal{D}'(U)$$

if

$$T_n(f) \to T(f) \qquad \text{for all } f \in \mathcal{D}(U).$$

If $f \in \mathcal{D}(u)$ and $g \in C^\infty(U)$ then $fg \in \mathcal{D}(U)$.

Definition 4.10. *Let $\phi \in C^\infty(U)$ and $T \in \mathcal{D}'(U)$. We define their product by*

$$(\phi T)(f) = T(\phi f)$$

and the derivative

$$(\partial_{x_j} T)(f) = T(-\partial_{x_j} f).$$

It is easy to see that the right hand side of the formulas defines a distribution. We can easily calculate Leibniz' formula in the form

$$\begin{aligned}\partial_{x_j}(\phi T)(f) = -T(\phi \partial_{x_j} f) &= T((\partial_{x_j}\phi) f) - T(\partial_{x_j}(\phi f)) \\ &= [(\partial_{x_j}\phi)T](f) + (\phi \partial_{x_j} T)(f)\end{aligned}$$

and the associative and distributive law:

$$\phi(\psi T) = \psi(\phi T).$$

Similarly the theorem of Schwarz holds

$$\partial_{x_j}\partial_{x_k} T = \partial_{x_k}\partial_{x_j} T.$$

Let $L^1_{loc}(U)$ be the set of measurable functions on U which are integrable on compact subsets. We say f_j converges to f in L^1_{loc} if $f_j|_K \to f|_K$ in $L^1(K)$ for all compact subsets K.

Definition 4.11. *We define $L^1_{loc}(U) \ni f \to T_f \in \mathcal{D}'(U)$ by*

$$T_f(\phi) = \int_U f\phi dm^d$$

for $\phi \in \mathcal{D}(U)$.

Lemma 4.12. *The map $L^1_{loc} \to \mathcal{D}'$ is linear, continuous and injective.*

Proof. Only injectivity has to be proven. After multiplying by a characteristic function of a ball we consider $f \in L^1(B)$. Suppose that

$$\int f\phi dx = 0$$

for all $\phi \in C_c^\infty$ supported in $B_1(0)$. Then

$$f * \phi(x) = 0$$

for all ϕ supported in $B_r(0)$ and $|x| < 1 - r$. But we have seen that there is such a sequence ϕ_j so that $f * \phi_j \to f$ in L^1. Then $f|_{B_{1-r}(0)} = 0$. This implies the full statement. $\square$

Similarly, any Radon measure μ on $U \subset \mathbb{R}^d$ defines a linear map from the continuous functions with compact support to $\mathbb{K}$, and we identify it with the restriction to $\mathcal{D}$.

Examples:

1. The Dirac measure δ_0
2. The Heaviside function $H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ satisfies

$$\partial_x H = \delta_0.$$

Firstly $H \in L^1_{loc}$. If $\phi \in \mathcal{D}(U)$

$$(\partial_x H)(\phi) = (\partial_x T_H)(\phi) = -T_H(\phi') = -\int_0^\infty H(x)\phi'(x)dx = \phi(0) = \delta_0(\phi).$$

3. Let

$$f = \begin{cases} 1/2 & \text{if } |x| < -t \\ 0 & \text{if } |x| \geq -t \end{cases}$$

Then

$$\begin{aligned}(\partial^2_{tt} - \partial^2_{xx})f(\phi) =& \frac{1}{2}\int_{-\infty}^0 \int_t^{-t} (\partial_t^2 - \partial_x^2)\phi(t,x)dxdt \\ =& \phi(0,0) = \delta_0(t,x),\end{aligned}$$

which is the contents of exercise 3 on sheet 8.

4. Let $d > 2$ and

$$g(x) = \frac{2}{d-2}\frac{\Gamma(\frac{d}{2})}{\pi^{d/2}}|x|^{2-d}$$

In *Einführung in die PDG* we have seen that

$$-\Delta g * \phi = \int g(x-y)(-\Delta\phi(y))dy = \phi(x).$$

Thus

$$-\Delta g = \delta_0.$$

[09.12.2016]

[14.12.2016]

Lemma 4.13. *The following identities hold*

$$\begin{aligned} T_{\phi\psi} &= \phi T_\psi \text{ for } \phi,\, \psi \in C(U), \\ T_{\partial_{x_j}\phi} &= \partial_{x_j} T_\phi \text{ for } \phi \in C^1(U). \end{aligned}$$

Proof. We use Fubini, integration by parts and the fact that $f \in \mathcal{D}(U)$ has compact support, to get

$$\begin{aligned} T_{\phi\psi}(f) &= \int_U \phi\psi f \, dm^d = (\phi T_\psi)(f), \\ T_{\partial_{x_j}\phi} f &= \int f \partial_{x_j} \phi \, dm^d = \int (-\partial_{x_j} f) \phi \, dm^d = (\partial_{x_j} T_\phi)(f). \end{aligned}$$

□

Definition 4.14. *Let $T \in \mathcal{D}'(U)$. We say that T vanishes near $x \in U$ if there exists $r > 0$ so that $T(f) = 0$ for all $f \in \mathcal{D}(U)$ with support in $B_r(x)$. We define the support of T as the complement of the points near which T vanishes.*

Lemma 4.15. *Let $\phi \in \mathcal{D}(U)$ and $T \in \mathcal{D}'(U)$ with disjoint supports. Then*

$$T(\phi) = 0.$$

Proof. Let K be the support of ϕ. Given $x \in K$ there exists r so that $T\psi = 0$ for every $\psi \in \mathcal{D}(U)$ supported in $B_r(x)$. Since K is compact there is a finite covering of such balls $B_{r_j}(x_j)$ with $1 \le j \le N$. We choose a partition of unity $\eta_j \in C^\infty(U)$ supported in $B_{r_j}(x_j)$ so that

$$\sum_{j=1}^N \eta_j(x) = 1$$

for $x \in K$. Then

$$T\phi = \sum_{j=1}^N T(\eta_j \phi) = 0$$

□

Definition 4.16. *Let $T \in \mathcal{D}'(\mathbb{R}^d)$ and $\phi \in \mathcal{D}(\mathbb{R}^d)$. We define their convolution by*

$$(\phi * T)(x) = T(\phi(x - \cdot)), \quad \forall x \in \mathbb{R}^d.$$

The righthand side denotes T acting on $\phi(x - y)$ as a function of y.

Lemma 4.17. *With the notation above, we know $\phi * T \in C^\infty(\mathbb{R}^d)$ and*

$$\partial_{x_j}(\phi * T) = (\partial_{x_j}\phi) * T = \phi * (\partial_{x_j} T).$$

If $\psi \in L^1(\mathbb{R}^d)$ then

$$\phi * T_\psi(x) = \phi * \psi(x).$$

Moreover, if $\operatorname{supp}\phi = K_1$ and $\operatorname{supp} T = K_2$ then

$$\operatorname{supp}\phi * T \subset K_1 + K_2 = \{x + y : x \in K_1 \text{ and } y \in K_2\}.$$

Proof. For $v \in \mathbb{R}^d$, we have to prove that

$$\frac{(\phi * T)(x+tv) - (\phi * T)(x)}{t} = T(\frac{1}{t}(\phi(x+tv-\cdot) - \phi(x-\cdot)))$$
$$\to T\left(\sum_{j=1}^d v_j(\partial_j\phi)(x-\cdot)\right) = \Big(\sum_{j=1}^d v_j\partial_j\phi\Big) * T(x)$$
$$\equiv -T\left(\sum_{j=1}^d v_j\partial_j(\phi(x-\cdot))\right) = \phi * \Big(\sum_{j=1}^d v_j\partial_j T\Big)(x).$$

This is a consequence of Lemma 4.8 and that the difference quotient

$$\frac{(\partial^\alpha\phi)(x+tv) - (\partial^\alpha\phi)(x)}{t} \to \sum_{j=1}^d v_j\partial_j(\partial^\alpha\phi)(x) \text{ as } t \to 0$$

uniformly in x since the support is compact.

The remaining properties are not hard to verify. (Exercise) $\square$

Remark 4.18. *We can equivalently define the convolution of $\phi \in \mathcal{D}(\mathbb{R}^d)$ and $T \in \mathcal{D}'(\mathbb{R}^d)$ as the distribution:*

$$(\phi * T)(f) = T(\tilde{\phi} * f), \quad \forall f \in \mathcal{D}(\mathbb{R}^d),$$

where $\tilde{\phi}(x) = \phi(-x)$.

In the same way we can easily define the convolution $\phi * T \in \mathcal{D}'(\mathbb{R}^d)$ when $\phi \in C^k(\mathbb{R}^d)$ has compact support. By an abuse of notation we write the convolution evaluated at x whenever it is defined, even if it is not defined on all of $\mathbb{R}^d$.

Definition 4.19. *Let $T \in \mathcal{D}'(\mathbb{R}^d)$ and let $S \in \mathcal{D}'(\mathbb{R}^d)$ with compact support. We define their convolution by*

$$(S * T)(\phi) = T(\tilde{S} * \phi)$$

for $\phi \in \mathcal{D}(\mathbb{R}^d)$. Here $\tilde{S}(\psi) = S(\tilde{\psi})$, $\quad \psi \in \mathcal{D}(\mathbb{R}^d)$.

It is an exercise to formulate and prove reasonable properties of the convolution of distributions.

Example 4.20.

Let $d > 2$ and

$$g(x) = \frac{2}{d-2}\frac{\Gamma(\frac{d}{2})}{\pi^{d/2}}|x|^{2-d} \in L^1_{loc}(\mathbb{R}^d).$$

Then

$$-\Delta g = \delta_0$$

and
$$-\Delta(g*\phi)=\delta_0*\phi=\phi.$$

We identify $\Delta\delta_0$ with the distribution
$$\Delta\delta_0(f)=\Delta f(0),\quad \forall f\in\mathcal{D}(\mathbb{R}^d).$$
Its support is $\{0\}$. Then
$$\phi*(\Delta\delta_0)(x)=\Delta\delta_0(\phi(x-\cdot))=\Delta\phi(x).$$
The same construction works for all differential operators with constant coefficients.

Lemma 4.21. *Suppose that $T_n\to T$ in $\mathcal{D}'(U)$ and that $K\subset U$ is compact. Then there exists k and C so that*
$$\sup_n|T_n(f)|\le C\|f\|_{C_b^k(U)}$$
for all $f\in X_K=\{f\in\mathcal{D}(U): suppf\subset K\}$ and
$$\sup\{|T_n(f)-T_m(f)|: f\in X_K, \|f\|_{C_b^k}\le 1\}\to 0$$
as $n,m\to\infty$.

Proof. The proof of the first part is the same is for Lemma 4.8. So, given K, there exist $k\ge 1$ and C so that for any n
$$|T_n(f)|\le C\|f\|_{C_b^{k-1}}.$$
Since K is compact and
$$C_c^k(K)\ni f\to(\partial^\alpha f)_{|\alpha|\le k}\in C_b(\Sigma_k\times K)=C_b(K;\mathbb{K}^{\#\Sigma_k})$$
is an isometry where Σ_k is the set of all multiindices of length at most k. By a multiple application of Theorem 3.34
$$\bar B_1(0)=\{f\in X_K|\|f\|_{C_b^1(K;\mathbb{K}^{\#\Sigma_{k-1}})}\le 1\}\subset C_b^1(K;\mathbb{K}^{\#\Sigma_{k-1}})$$
is compact in $C_c^{k-1}(K)$. Let $\varepsilon>0$. Then there exist finitely many functions $f_m\in C_b^k(K)$ so that the ε balls in C_b^{k-1} centered at f_m cover $\bar B_1(0)$. We may assume that they are in C_c^∞ by Lemma 4.6. There exists n_0 so that
$$|(T_n-T)(f_m)|<\varepsilon$$
if $n\ge n_0$. Then, for any $f\in\bar B_1(0)$ there exists f_m such that $\|f-f_m\|_{C_b^{k-1}}<\varepsilon$ and hence for $n\ge n_0$ there hold
$$|(T_n-T)f|\le|(T_n-T)(f_m)|+|T_n(f-f_m)|+|T(f-f_m)|\le\varepsilon+2C\varepsilon.$$
□

Theorem 4.22. *Let $U \subset \mathbb{R}^d$ be open. Then $\mathcal{D}(U) \subset \mathcal{D}'(U)$ is dense.*

Proof. There are several steps.
Step 1: Distributions with compact support are dense. We choose a sequence $\phi_j \in \mathcal{D}(U)$ so that

$$\partial^\alpha \phi_j \to \partial^\alpha 1$$

on compact subsets. Then $\phi_j T$ has compact support and

$$(\phi_j T)(g) = T(\phi_j g) \to Tg$$

for all $g \in \mathcal{D}(U)$.

Step 2: Construction of the ϕ_j. Let K_j be a monotone sequence of compact sets so that K_j is contained in the interior of K_{j+1} and $U = \bigcup K_j$. Then for any j there exists $r_j > 0$ so that

$$\min\{\operatorname{dist}(K_{j-1}, \mathbb{R}^d \backslash K_j), \operatorname{dist}(K_j, \mathbb{R}^d \backslash K_{j+1})\} \geq r_j > 0.$$

Let $\varphi \in C_c^\infty(B_1(0))$ be radial with $\int \varphi dx = 1$ and let $\varphi_r(x) = r^{-d}\varphi(x/r)$. Then

$$\phi_j = \varphi_{r_j} * \chi_{K_j} \in C^\infty,$$
$$\operatorname{supp} \phi_j \subset K_{j+1}, \quad \phi_j = 1 \text{ on } K_{j-1}.$$

Step 3 Let $T \in \mathcal{D}'(U)$ have compact support. Then, for r small and $g \in \mathcal{D}(U)$

$$\varphi_r * T(g) = T(\tilde{\varphi}_r * g) \to T(g), \quad r \to 0$$

by an abuse of notation. □

The same argument gives Lemma 4.6.

Lemma 4.23. *Suppose that U is connected and $\partial_j T = 0$, $j = 1, \cdots, d$. Then there eixsts a constant c so that $T = T_c$.*

Proof. Exercise. □

[14.12.2016]

[16.12.2016]

4.3 Schwartz functions and tempered distributions

We briefly cover the definition of Schwartz functions and tempered distributions, which are the proper frame work for the Fourier transform.

Definition 4.24. *The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ consists of all Schwartz functions, which are functions f so that for all multiindices α, β*

$$\|x^\alpha \partial^\beta f\|_{sup} < \infty.$$

We say $f_j \to f$ as Schwartz functions if for all multiindices α and β $x^\alpha \partial^\beta f_j \to x^\alpha \partial^\beta f$ uniformly.

Remark 4.25. *It is easy to see that if $f \in \mathcal{S}(\mathbb{R}^d)$, then for any N there exists C_N such that $|f(x)| \le C_N(1+|x|)^{-N}$, and hence $f \in L^p(\mathbb{R}^d)$, for any $p \in [1,\infty]$. So is $\partial^\alpha f$ for any multiindex α.*

Roughly speaking, the Schwartz functions have two properties: they have infinite bounded derivatives and they decay fast at infinity. Recalling $\widehat{\partial_{x_j} f}(\xi) = 2\pi i \xi_j \hat{f}(\xi)$ and $\widehat{x_j f}(\xi) = \frac{i}{2\pi}\partial_{\xi_j}\hat{f}(\xi)$, the Fourier transform should work well in the framework of Schwartz space and tempered distributions (see below).

Since $\eta_n f \to f$ in $\mathcal{S}(\mathbb{R}^d)$, $\eta_n(x) = \eta(n^{-1}x)$ where $\eta \in C_c^\infty(B_2(0))$ takes value 1 on $B_1(0)$, the inclusion $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ is dense.

Lemma 4.26. *Let f be a Schwartz function.*

1. *If α is a multiindex then $\partial^\alpha f \in \mathcal{S}(\mathbb{R}^d)$.*
2. *If $g \in C^\infty$ and for any multiindex α there exist $c_{|\alpha|}$ and $\kappa_{|\alpha|}$ so that*
$$|\partial^\alpha g| \le c_{|\alpha|}(1+|x|)^{\kappa_{|\alpha|}}$$
then $gf \in \mathcal{S}(\mathbb{R}^d)$.
3. *If $g \in C(\mathbb{R}^d)$ satisfies for any multiindex α*
$$\|x^\alpha g(x)\|_{sup} < \infty,$$
then $g * f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. The first property follows from the definition. By the first property, in order to prove the second property it suffices to show
$$\sup |x^\alpha (\partial^\beta g) f| < \infty,$$
which follows from the definition.

Since
$$\partial^\alpha (g * f) = g * \partial^\alpha f$$
and since $\partial^\alpha f$ is Schwartz, by the first property the proof of the third property is reduced to bounding
$$\|x^\alpha (g * f)\|_{sup}.$$
We observe that
$$x_j(g * f) = (x_j g) * f + g * (x_j f)$$
and the claim follows by induction on the length of α. □

Definition 4.27. *We define $d : \mathcal{S} \times \mathcal{S} \to [0,\infty)$ by*
$$d(f,g) = \sup_k 2^{-k} \min\{1, \sup_{|\alpha|+|\beta|=k} \|x^\alpha \partial^\beta (f-g)\|_{sup}\}.$$

Lemma 4.28. *The expression $d(f,g)$ defines a metric on $\mathcal{S}$ which turns it into a complete metric space.*

Proof. Easy exercise. $\square$

Definition 4.29. *A tempered distribution is a continuous linear map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathbb{K}$. We denote the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. We say that $T_j \to T$ in $\mathcal{S}'(\mathbb{R}^d)$ if $T_j(\phi) \to T(\phi)$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$.*

Lemma 4.30. *Let $T \in \mathcal{S}'(\mathbb{R}^d)$. Then there exist k and c so that*

$$|T\phi| \le c \sup_{|\alpha|+|\beta|\le k} \|x^\alpha \partial^\beta \phi\|_{sup}.$$

If $T_n \to T$ in $\mathcal{S}'(\mathbb{R}^d)$ then there exist C and k so that

$$\sup_n |T_n\phi| \le C \sup_{|\alpha|+|\beta|\le k} \|x^\alpha \partial^\beta \phi\|_{sup}$$

and

$$\frac{|T_n(\phi) - T\phi|}{\sup_{|\alpha|+|\beta|\le k} \|x^\alpha \partial^\beta \phi\|_{sup}} \to 0.$$

Proof. The proof is similar as that of Lemma 4.21. The existence of C and k follows from the idea of the proof of Banach-Steinhaus theorem. The convergence result follows from the compactness of the ball $\{\phi \in \mathcal{S}(\mathbb{R}^d) \mid \sup_{|\alpha|+|\beta|\le k} \|x^\alpha \partial^\beta \phi\|_{sup} \le 1\}$ in the space $\{\phi \in \mathcal{S}(\mathbb{R}^d) \mid \sup_{|\alpha|+|\beta|\le k-1} \|x^\alpha \partial^\beta \phi\|_{sup} < +\infty\}$, which is easy to see if we notice that

$$\sup_{|\alpha|+|\beta|\le k-1} \|x^\alpha \partial^\beta \phi\|_{sup} \le R^{-1} \sup_{|\alpha|+|\beta|\le k} \|x^\alpha \partial^\beta \phi\|_{sup((B_R(0))^C)} + R^{k-1} \|\phi\|_{C_b^{k-1}(B_R(0))}$$

and we can choose R big enough. $\square$

Remark 4.31. *We define the derivative and the multiplication by a smooth function with controlled derivatives for a tempered distribution as we did it for distributions. Similarly, since compactly supported distributions S can act on Schwartz functions f, we can define the convolution $(S * f)(x) \in \mathcal{S}(\mathbb{R}^d)$. We then can define the convolution of a tempered distribution with Schwartz functions and with compactly supported distributions.*

Let $1 \le p \le \infty$. There are the embeddings

$$\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d).$$

The embeddings are dense if $p < \infty$.

4.4 Sobolev spaces: Definition

Definition 4.32. *Let $U \subset \mathbb{R}^d$ be open, $k \in \mathbb{N}$ and $1 \le p \le \infty$. The Sobolev space $W^{k,p}(U) \subset L^p(U)$ is the set of all $L^p(U)$ functions, so that for all multiindices α of length at most k there exists $f_\alpha \in L^p(U)$ so that*

$$\partial^\alpha T_f = T_{f_\alpha}.$$

We define (identifying T_f and f and $\partial^\alpha T_f$ with f_α by an abuse of notation)

$$\|f\|_{W^{k,p}} = \left(\sum_{|\alpha| \le k} \|\partial^\alpha f\|^p_{L^p(U)} \right)^{\frac{1}{p}}$$

with the usual modification if $p = \infty$.

We have

$$g = \partial_{x_j} f, \quad f, g \in L^p(U)$$

if and only if

$$\int g \phi dm^d = -\int f \partial_{x_j} \phi dm^d$$

for all $\phi \in \mathcal{D}(U)$.

Lemma 4.33. *Let $g \in C^k_b(U)$ and $f \in W^{k,p}(U)$. Then $gf \in W^{k,p}(U)$.*

Proof: Easy exercise.

Definition 4.34. *Let $k \in \mathbb{N}$, $1 \le p < \infty$. We define $W^{k,p}_0(U)$ as the closure of $C^\infty_c(U)$ with respect to the norm $\|\cdot\|_{W^{k,p}(U)}$. If $V \subset U$ the extension defines a canonical (nonsurjective) isometry from $W^{k,p}_0(V)$ to $W^{k,p}_0(U)$.*

Theorem 4.35. *Let $k \in \mathbb{N}$ and $1 \le p \le \infty$. The Sobolev space $W^{k,p}(U)$ is a Banach space. If $V \subset U$ then the restriction defines a map of norm 1 from $W^{k,p}(U)$ to $W^{k,p}(V)$. Moreover $W^{k,p}_0(\mathbb{R}^d) = W^{k,p}(\mathbb{R}^d)$ if $1 \le p < \infty$.*

Proof. Let $\Sigma_k = \{\alpha : |\alpha| \le k\}$. There is an obvious isometry

$$W^{k,p}(U) \ni f \to (f_\alpha)_{|\alpha| \le k} \in L^p(U \times \Sigma_k).$$

Let f_j be a Cauchy sequence in $W^{k,p}(U)$ with limit $f \in L^p(U)$. Then $\partial^\alpha f_j \to f_\alpha$ in $L^p(U)$ and in $\mathcal{D}'(U)$. It is easy to check that $f_\alpha = \partial^\alpha f$ in $\mathcal{D}'(U)$: for any $\phi \in \mathcal{D}(U)$,

$$T_{f_\alpha}(\phi) = \lim_{j \to \infty} (-1)^{|\alpha|} \int_U f_j \partial^\alpha \phi \, dm^d = (-1)^{|\alpha|} \int_U f \partial^\alpha \phi \, dm^d = T_{\partial^\alpha f}(\phi).$$

Thus $f \in W^{k,p}(U)$ and $W^{k,p}(U)$ is complete and we can identify $W^{k,p}(U)$ with a closed subspace of $L^p(U \times \Sigma_k)$.

The restriction map with norm ≤ 1 follows from the definition, and the norm is indeed 1 since

$$\|f\|_{W^{k,p}(V)} = \|\widetilde{f}\|_{W^{k,p}(U)}, \quad f \in W_0^{k,p}(V),$$

and $\widetilde{f} \in W_0^{k,p}(U)$ is the trivial extension of f by 0.

Now let $1 \leq p < \infty$. Density of $\mathcal{D}(\mathbb{R}^d) \subset W^{k,p}(\mathbb{R}^d)$ follows by the same argument as in the proof of Theorem 4.22. □

[16.12.2016]

[21.12.2016]

Definition 4.36. *Let $k \in \mathbb{N}$ and $1 < p < \infty$. We define $W^{-k,p}(U) = (W_0^{k,p'}(U))^*$.*

Lemma 4.37. *The map $J\colon L^p(U \times \Sigma_k) \to W^{-k,p}(U)$ defined by*

$$J((f_\alpha))(u) = \sum_{|\alpha|\leq k} \int_U f_\alpha \partial^\alpha u \, dm^d, \quad \forall u \in W_0^{k,p'}(U)$$

has norm 1.

Proof. Exercise. □

Lemma 4.38. *Let $U \subset \mathbb{R}^d$ be bounded and open and $k \in \mathbb{N}$. We assume that $f \in C_b^k(U)$ and its derivatives of order up to $k-1$ extend to continous functions in $\bar{U}$ which vanish at ∂U. Then $f \in W_0^{k,p}(U)$ for $1 \leq p < \infty$.*

Proof. We proceed as in Theorem 4.22. Let $K_j = \{x \in U : \operatorname{dist}(x, \partial U) \geq 2^{-j}\}$. We extend χ_{K_j} by 0 to $\mathbb{R}^d$ and convolve it with a smooth function (say $\varphi_{2^{-j-1}}$) of integral 1 supported in $B_{2^{-j-1}}(0)$, to obtain $\eta_j \in C_c^\infty(U)$. Then $\operatorname{supp} \eta_j \subset \{x \in U : \operatorname{dist}(x, \partial U)\} \leq 2^{-j-1}$ and $\eta_j(x) = 1$ for $\operatorname{dist}(x, \partial U) \geq 2^{1-j}$ and

$$|\partial^\alpha \eta_j| \leq c(|\alpha|) 2^{|\alpha| j}.$$

Since for $|\alpha| \leq k$, by the Taylor formula,

$$|\partial^\alpha f(x)| \leq c \operatorname{dist}(x, \partial U)^{k-|\alpha|}.$$

Thus the sequence $\eta_j f$ is uniformly bounded in $C_b^k(U)$, and hence in $W^{k,p}(U)$. Moreover

$$\partial^\alpha(\eta_j f) \to \partial^\alpha f$$

for every $x \in U$ and by dominated convergence

$$\eta_j f \to f \qquad \text{in } W^{k,p}(U).$$

We complete the proof by regularizing $\eta_j f$ as in Lemma 4.6. □

The cofactor matrix cof A of an $n \times n$ matrix has as (i,j) entry $(-1)^{i-j}$ times the determinant of the $(n-1)\times(n-1)$ matrix obtained from A by removing the ith row and the jth column. It is the same as the partial derivative of $\det(A)$ with respect to the (i,j)th entry. Then (linear algebra)

$$A^T \operatorname{cof} A = \det A 1_{n\times n} \tag{4.1}$$

Lemma 4.39. *Let $U \subset \mathbb{R}^d$ be open and $\phi \in C^2(U;\mathbb{R}^d)$. Then*

$$\sum_{j=1}^d \partial_{x_j} \operatorname{cof}(D\phi)_{ij} = 0.$$

Proof. ¿From (4.1) with $A = D\phi$,

$$\begin{aligned}
\partial_{x_i} \det(D\phi) =& \sum_{j=1}^d \delta_{ij}\partial_{x_j} \det(D\phi)\\
=&\delta_{ij} \sum_{j=1}^d \sum_{k,m=1}^d (\operatorname{cof} D\phi)_{km} \partial^2_{x_m x_j}\phi^k\\
=& \sum_{j',k=1}^d (\partial_{x_i}\partial_{x_{j'}}\phi^k)(\operatorname{cof} D\phi)_{kj'} + \sum_{j,k=1}^d \partial_{x_i}\phi^k \partial_j \operatorname{cof}(D\phi)_{kj}\\
=& \sum_{j,k=1}^d \big(\partial^2_{x_i x_j}\phi^k (\operatorname{cof} D\phi)_{kj} + \partial_{x_i}\phi^k\partial_{x_j}(\operatorname{cof} D\phi)_{kj}\big).
\end{aligned}$$

which implies

$$\sum_{k=1}^d \partial_{x_i}\phi^k \left(\sum_{j=1}^d \partial_{x_j}(\operatorname{cof} D\phi)_{kj}\right) = 0.$$

This implies the claim if $\det D\phi \neq 0$. If $\det D\phi(x) = 0$ we apply the reasoning to $\phi + \varepsilon x$ and send ε to 0. $\square$

Lemma 4.40. *If $\phi : V \to U$ is a C^k_b diffeomorphism (bounded derivatives of ϕ and ϕ^{-1}) then there exists $C > 1$ so that*

$$\|f\circ\phi\|_{W^{k,p}(V)} \le C\|f\|_{W^{k,p}(U)}.$$

Moreover the chain rule holds

$$\partial_{y_j}(f\circ\phi) = \sum_{k=1}^d (\partial_{x_k} f\circ\phi)\partial_{y_j}\phi_k.$$

Proof. The first claim follows from the chain rule and the transformation formula. We prove the chain rule for a smooth diffeomorphism. The general case follows by approximating the diffeomorphism and taking limits. We write $\psi \circ \phi(y) = \tilde{\psi}(y)$. Then

$$
\begin{aligned}
-\int_V (f\circ\phi(y))\partial_{y_j}\tilde{\psi}(y)dm^d(y) &= -\int_U f(x)\sum_{k=1}^d \frac{\partial x_k}{\partial y_j}\partial_{x_k}\psi \det(D\phi^{-1})dm^d(x)\\
&= -\sum_{k=1}^d \int_U f(x)\partial_k(\frac{\partial x_k}{\partial y_j}\psi\det(D\phi^{-1}))dm^d(x)\\
&\quad +\sum_{k=1}^d \int_U f(x)\psi\partial_{x_k}(\frac{\partial x_k}{\partial y_j}\det(D\phi^{-1}))dm^d(x)
\end{aligned}
$$

The second term vanishes by Lemma 4.39 and we continue, assuming that ϕ is smooth (which requires an approximation argument)

$$
= \sum_{k=1}^d \int_U (\partial_{x_k} f)\frac{\partial x_k}{\partial y_j}\det(D\phi^{-1})dm^d(x).
$$

The chain rule follows by another application of the transformation formula. □

4.5 (Whitney) extension and traces

Definition 4.41. *Let $U \subset \mathbb{R}^d$ be open, bounded and connected. We say that U is a Lipschitz domain if there exist a continuous vector field ν on ∂U and a Lipschitz continuous function ρ and $c > 0$ so that $\partial U = \rho^{-1}(\{0\})$ and*

$$
\rho(x+t\nu) - \rho(x+s\nu) \ge t - s
$$

for $x \in \partial U$ and $-c < s < t < c$.

Examples:

1. Bounded connected open sets with C^1 boundary.
2. Let $h : \mathbb{R}^{d-1} \to \mathbb{R}$ be Lipschitz continuous with Lipschitz constant L. The set below the graph is not compact, but the other conditions are satisfied with $\rho(x) = x_d - h(x_1, \cdots, x_{d-1})$ and $\nu = e_d$.

Theorem 4.42 (Whitney)**.** *Let $1 \le p < \infty$ and $U \subset \mathbb{R}^d$ be a Lipschitz domain. Then there exists a linear extension map*

$$
W^{k,p}(U) \to W^{k,p}(\mathbb{R}^d).
$$

Proof. We prove the theorem under the stronger assumption that ∂U is a C^k manifold. By use of rotation, compactness and partition of unity, it suffices to consider the extension problem for $U = \{x : x_d < \psi(x_1, \dots, x_{d-1})\}$ where ψ is a function in C^k_b. By Lemma 4.40 we can choose $\phi(x) = (x_1, \dots, x_{d-1}, x_d - \psi(x_1, x_2, \dots x_{d-1}))$ to reduce the problem to extending Sobolev functions on the lower half space $V = \{x | x_d \le 0\}$. Let f be defined on V. We make the Ansatz

$$F(x) = \begin{cases} f(x) & \text{if } x_d \le 0 \\ \sum\limits_{j=1}^{k+1} a_j f(x_1, \dots x_{d-1}, -jx_d) & \text{if } x_d > 0. \end{cases}$$

If $f \in C^k(V)$ we want to choose the a_j so that $\partial^\alpha F$ is continuous for $|\alpha| \le k$. It clearly suffices to do this for $\partial^j_{x_d}$ for $0 \le j \le k$ when $\{x_d = 0\}$, which leads to the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 3 & \dots & k+1 \\ 1^2 & 2^2 & 3^2 & \dots & (k+1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1^k & 2^k & 3^k & \dots & (k+1)^k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{k+1} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ \dots \\ (-1)^k \end{pmatrix}.$$

The Vandermonde matrix is invertible and we can solve this system. The coefficients a_j hence exist and depend only on k. Then

$$\|F\|_{W^{k,p}(\mathbb{R}^d)} \le C \|f\|_{W^{k,p}(V)}.$$

Now we would like to pass from the assumption $f \in C^k(U)$ to $f \in W^{k,p}(U)$. We define the extension F in the same way. Then we have to prove that for $|\alpha| \le k$ the distributional derivative is given by the distribution defined by the distributional derivatives on both sides. By an application of the theorem of Fubini and using distributional derivatives in $d-1$ variables this is true for all α with $\alpha_d = 0$. Again by Fubini it suffices to consider $d = 1$. Let $1 \le \kappa \le k$ and $\phi \in \mathcal{D}(\mathbb{R})$. Then, with

$$\partial^\kappa F = \begin{cases} \partial^\kappa f & \text{if } x < 0 \\ \sum\limits_{j=1}^{k+1} a_j \frac{d^\kappa}{dx^\kappa}(f(-jx)) & \text{if } x > 0 \end{cases}$$

we obtain

$$\begin{aligned}(-1)^\kappa \int_{\mathbb{R}} F \frac{d^\kappa}{dx^\kappa}\phi dx =&(-1)^\kappa \int_{-\infty}^0 f \frac{d^\kappa}{dx^\kappa}\phi dx + (-1)^\kappa \sum_{j=1}^{k+1} a_j \int_0^\infty f(-jx)\frac{d^\kappa}{dx^\kappa}\phi(x)dx\\
=&\int_{-\infty}^0 f\frac{d^\kappa}{dx^\kappa}\Big[(-1)^\kappa\phi + \sum_{j=1}^{k+1} j^{\kappa-1}a_j\phi(-x/j)\Big]dx\\
=&\int_{-\infty}^0 \frac{d^\kappa}{dx^\kappa} f(x)[\phi(x) - \sum_{j=1}^{k+1} a_j(-j)^{\kappa-1}\phi(-x/j)]dx\\
=&\int_{\mathbb{R}} \frac{d^\kappa}{dx^\kappa}F\phi(x)dx.\end{aligned}$$

In last equality we observe that

$$\frac{d^{\kappa'}}{dx^{\kappa'}}(\phi(x) - \sum_{j=1}^{k+1} a_j(-j)^{\kappa-1}\phi(-x/j)) = 0$$

for $x = 0$ and $\kappa' < \kappa$ by the definition of the a_j: Indeed, we observe that

$$\Big(1 - \sum_{j=1}^{k+1} a_j(-j)^{\kappa-\kappa'-1}\Big)\phi^{(\kappa')}(0) = 0.$$

Hence it is in $W_0^{\kappa,p}((-\infty,0))$ by Lemma 4.38 and we can approximate it by functions in $\mathcal{D}((-\infty,0))$. For those we can move the derivatives to f. $\square$

Corollary 4.43. *Suppose that U is a open, bounded with Lipschitz boundary, $k \in \mathbb{N}$ and $1 \le p < \infty$. Then the restrictions of $C_0^\infty(\mathbb{R}^d)$ functions is dense in $W^{k,p}(U)$.*

[21.12.2016]

[23.12.2016]

Theorem 4.44 (Traces)**.** *Let U be a bounded domain with C^1 boundary and let $f \in W^{1,p}(U)$, $1 \le p \le \infty$. Then there is a unique trace $g \in L^p(\partial U)$ so that*

$$\int_{\partial U} \sum_{j=1}^d F^j\nu^j g d\mathcal{H}^{d-1} = \int_U f\sum_{j=1}^d \partial_{x_j}F^j dm^d + \int_U \sum_{j=1}^d \partial_{x_j} f F^j dm^d, \qquad (4.2)$$

where $F^j \in C^1(\overline{U})$ and ν denotes the outer normal vector of ∂U. It satisfies

$$\|g\|_{L^p(\partial U)} \le c\|f\|_{L^p(U)}^{\frac{p-1}{p}}\|Df\|_{L^p}^{\frac{1}{p}}, \quad \|Df\|_{L^p} := \|\,|(\partial_{x_j}f)|\,\|_{L^p}.$$

Proof. If $f \in C^1(\overline{U})$ then $g = f|_{\partial U}$ has the desired properties. By a partition of unity we may assume that U is below the graph of a C^1_b function ϕ, and that f has compact support. Then, if $p < \infty$,

$$\begin{aligned}\|g\|^p_{L^p(\partial U)} = \int |g|^p dm^{d-1} = & \int_{\mathbb{R}^{d-1}} |g|^p (1+|\nabla\phi|^2)^{\frac12} dm^{d-1}(x') \\ =& p \operatorname{Re} \int_U (1+|\nabla'\phi(x')|^2)^{\frac12} |f|^{p-1} \bar f \partial_{x_d} f dm^d \\ \le & p \sup(1+|\nabla'\phi|^2)^{\frac12} \|f\|^{p-1}_{L^p} \|\partial_{x_d} f\|_{L^p}.\end{aligned}$$

We approximate f by smooth functions, and go to the limit. Then we obtain a function g at the boundary so that (4.2) holds. We fix a vector field F so that $F \cdot \nu \ge 0$ at the boundary. Let $h \in C^1(\partial U)$ and it has an extension to $C^1(\overline{U})$, which we denote again by h. We apply (4.2) with $\tilde F = hF$. Then the right hand side of (4.2) (with F replaced by $\tilde F$) determines

$$\int_{\partial U} (F\cdot \nu) h g d\mathcal{H}^{d-1}$$

for $h \in C^1(\partial U)$. This determines g uniquely. The case $p = \infty$ is simpler.

□

4.6 Finite differences

Here we want to relate the analogue of finite differences to Sobolev functions.

Theorem 4.45. *a) Let $1 < p \le \infty$. If $f \in L^p$ and*

$$\sup_h \frac{\|f(.+h) - f\|_{L^p(\mathbb{R}^d)}}{|h|} \le C \tag{4.3}$$

then $f \in W^{1,p}$ and

$$\|\partial_{x_j} f\|_{L^p} \le \sup_{t\ne 0} \Big\| \frac{f(.+te_j) - f}{t} \Big\|_{L^p(\mathbb{R}^d)} \le C.$$

b) Now let $1 \le p \le \infty$ and $f \in W^{1,p}$. Then

$$\frac{f(.+te_j) - f(.)}{t} \to \partial_j f \qquad \text{in} \left\{ \begin{array}{ll} L^p & \text{if } p < \infty \\ \mathcal{D}' & \text{if } p = \infty \end{array}\right.$$

Proof. Suppose that (4.3) holds. Then

$$\lim_{t\to 0} \frac{f(x+te_1) - f(x)}{t} \to \partial_{x_1} f$$

as distribution:

$$T_{f_t}(\phi) = \frac1t \int (f(x+te_1) - f(x))\phi(x) dm^d = T_f\Big(\frac1t(\phi(x-te_1) - \phi(x))\Big)$$

and in $\mathcal{D}(\mathbb{R}^d)$

$$\frac{1}{t}(\phi(x-te_1)-\phi(x)) \to -\partial_{x_1}\phi(x).$$

The difference quotient defines an element in $L^{\frac{p}{p-1}}(\mathbb{R}^d)^*$. It is bounded by C uniformly with respect to t. Then also $\partial_{x_1} f$ defines an element in $(L^{\frac{p}{p-1}})^*$ with norm at most C. By the representation theorem there exists $\partial_{x_1} f \in L^p$. This proves the first direction and

$$\|\partial_{x_j} f\|_{L^p(\mathbb{R}^d)} \le \liminf_{t\to 0} \Big\| \frac{f(te_j+.)-f}{t} \Big\|_{L^p}.$$

Consider now $f \in W^{1,p}(U)$, $1 \le p \le \infty$ and we would like to show (4.3). By the fundamental theorem of calculus and Minkowski's inequality

$$\|f(.+h)-f(.)\|_{L^p(\mathbb{R}^d)} = \| \int_0^1 \sum_{j=1}^d h_j \partial_j f(.+sh) ds \|_{L^p(\mathbb{R}^d)} \le |h| \| |Df| \|_{L^p}$$

for $C^1 \cap W^{1,p}$ functions. Density completes the argument for $p < \infty$. For $p = \infty$ we use that by the previous argument

$$\begin{aligned} \Big| \int (f(x+h)-f(x))\phi(x) dm^d(x) \Big| =& \Big| \int_0^1 \int_{\mathbb{R}^d} \sum_{j=1}^d \partial_{x_j} f(x+th) h_j \phi(x) dm^d(x) dt \Big| \\ \le& |h| \| |\nabla f| \|_{L^\infty} \|\phi\|_{L^1} \end{aligned}$$

and hence

$$\| \frac{f(x+h)-f(x)}{|h|} \|_{L^\infty} \le \| |\nabla f| \|_{L^\infty}.$$

□

Corollary 4.46. *[Poincaré inequality] Let $U \subset \{x \in \mathbb{R}^d : a < x_1 < b\}$ and $f \in W^{1,p}_0(U)$. Then*

$$\|f\|_{L^p(U)} \le |b-a| \| |\nabla f| \|_{L^p(U)}.$$

Proof. Extend f by 0 to $\mathbb{R}^d$ and apply the previous theorem. □

We define

$$f_B = \fint_B f dm^d = (m^d(B))^{-1} \int_B f dm^d.$$

Lemma 4.47 (Poincaré inequality on ball)**.** *If $1 \le p \le \infty$ and $f \in W^{1,p}(B_R(0))$ then*

$$\|f - f_{B_R(0)}\|_{L^p(B_R(0))} \le 2^{\frac{d}{p}} R \| |Df| \|_{L^p(B_R(0))}.$$

Proof. It suffices to consider $R = 1$ by replacing f by $f(x/R)$. We calculate again by Minkowski's and Jensen's inequality

$$\begin{aligned}
&(m^d(B_1(0))^p \int_{B_1(0)} |f - f_{B_1(0)}|^p dm^d(x) \\
=& \int_{B_1(0)} \left| \int_{B_1(0)} (f(x) - f(y)) dm^d(y) \right|^p dm^d(x) \\
\le& \int\int_{B_1(0)\times B_1(0)} |f(x) - f(y)|^p dm^d(x) dm^d(y) \\
=& \int_{B_1(0)\times B_1(0)} \left| \int_0^1 \nabla f(x + t(y-x)) dt \right|^p |x-y|^p dm^d(x) dm^d(y) \\
\le& 2^p \int_0^1 \int_{B_1(0)\times B_1(0)} |\nabla f(x + t(y-x))|^p dm^d(x) dm^d(y) dt \\
=& 2^p \cdot 2 \int_0^{\frac{1}{2}} \int_{B_1(0)\times B_1(0)} |\nabla f(x + t(y-x))|^p dm^d(x) dm^d(y) dt
\end{aligned}$$

where we used symmetry in x and y in the last equality. However, if $y \in B_1(0)$ and $0 \le t \le \frac{1}{2}$ then

$$\int_{B_1(0)} |\nabla f(x + t(y-x))|^p dm^d(x) \le 2^d \int_{B_1(0)} |\nabla f(x)|^p dm^d(x)$$

by the transformation formula. □

[23.12.2016]

[11.01.2017]

There has been an omission in the proof of Theorem 4.45: We have not shown that the difference quotient converges in L^p to the derivative - only as a distribution.

4.7 Sobolev inequalities and Morrey's inequality

Lemma 4.48. *Let $f \in C_0(\mathbb{R})$, $f' \in L^1$. Then the Sobolev inequality*

$$\|f\|_{sup} \le \frac{1}{2}\|f'\|_{L^1} \tag{4.4}$$

holds. If $f' \in L^p$, $1 \le p \le \infty$ then every point is a Lebesgue point and

$$\sup_{x\ne y} \frac{|f(x) - f(y)|}{|x-y|^{1-\frac{1}{p}}} \le \|f'\|_{L^p}. \tag{4.5}$$

Proof. It suffices to prove the estimate for smooth functions with compact support. The inequality (4.4) is a consequence of the fundamental theorem of calculus:

$$f(x) = \int_{-\infty}^x f'(y)dy = -\int_x^\infty f'(y)dy.$$

and we derive (4.5) by Hölder's inequality

$$|f(x) - f(y)| \le \int_x^y |f'(z)|dz \le \|f'\|_{L^p} \|\chi_{[x,y]}\|_{L^{\frac{p}{p-1}}} = |x-y|^{1-\frac{1}{p}} \|f'\|_{L^p}.$$

This proof applies to $f \in C^1$. The general statement follows as in Morrey's inequality below. □

The Sobolev inequality and Morrey's inequality are the versions of these inequalities (4.4), (4.5) in higher space dimension.

Theorem 4.49 (Morrey)**.** *Let U be open. Suppose that $p > d$ and $\tilde f \in W^{1,p}(U)$. Every point is a Lebesgue point and the canonnical representative f is continuous. There exists c depending on p and d so that the following is true: Let $x, y \in U$ with*

$$|x - y| < \operatorname{dist}(x, \mathbb{R}^d \backslash U).$$

Then

$$|f(x) - f(y)| \le c|x-y|^{1-\frac{d}{p}} \| |\nabla f| \|_{L^p(B_{|x-y|}(x))}.$$

Proof. The inequality follows from

$$\left| f(y) - f_{B_R(x_0)} \right| \le C R^{1-\frac{d}{p}} \| |\nabla f| \|_{L^p(B_R(x_0))} \tag{4.6}$$

for $|y - x_0| < R$ (and Lebesgue points y) with a constant $C(p,d)$ which is bounded as $p \to \infty$ by

$$|f(y) - f(x_0)| \le \left| f(y) - f_{B_R(x_0)} \right| + \left| f(x_0) - f_{B_R(x_0)} \right|$$

and two applications of (4.6) if y and x_0 are Lebesgue points.

It remains to prove (4.6) and to prove that every point is a Lebesgue point. We have for $B_{R/2}(y) \subset B_R(x_0)$

$$\begin{aligned}
&\left| (m^d(B_R))^{-1} \int_{B_R(0)} f(x_0 + z) - f(y + z/2) dm^d(z) \right| \\
= &\left| (m^d(B_R))^{-1} \int_{B_R(0)} \int_0^1 \sum_{j=1}^d (x_0 + z/2 - y)_j \partial_{x_j} f(x_0 + (1 - t/2)z + t(y - x_0)) dt dm^d(z) \right| \\
\le &R 2^d (m^d(B_R))^{-1} \int_{B_R(x_0)} |\nabla f| dm^d(x) \\
\le &2^d (m^d(B_1(0)))^{-\frac{1}{p}} R^{1-\frac{d}{p}} \|\nabla f\|_{L^p(B_R(x_0))}
\end{aligned}$$

first for smooth functions, and then by approximation for Sobolev functions. By a geometric series and an iterative application of the above inequality with $(x_0, y) = (x_{j-1}, x_j)$:

$$
\begin{aligned}
|f(y) - f_{B_R(x_0)}| = & \left| \sum_{j=1}^{\infty} f_{B_{2^{-j}R}(x_j)} - f_{B_{2^{1-j}R}(x_{j-1})} \right| \\
\leq & \sum_{j=1}^{\infty} |f_{B_{2^{-j}R}(x_j)} - f_{B_{2^{1-j}R}(x_{j-1})}| \\
\leq & cR^{1-\frac{d}{p}} \sum_{j=1}^{\infty} 2^{-j(1-\frac{d}{p})} \| |\nabla f| \|_{L^p(B_R(x_0))}
\end{aligned}
$$

provided y is a Lebesgue point and for a suitable converging sequence of x_j. This however follows from the convergence for every point. □

Theorem 4.50 (Rademacher)**.** *I) Lipschitz continuous functions are almost everywhere differentiable.*

II) Functions in $W^{1,p}(U)$ with $p > d$ are almost everywhere differentiable. The derivative almost everywhere is the same as the weak derivative.

Proof. Part I is an exercise and we prove Part II. Let $f \in W^{1,p}$. By Morrey's theorem 4.49 we know that every point is a Lebesgue point and there is a uniformly continuous representative. By Theorem 3.51 there exists a set A whose complement has zero measure so that every $x \in A$ is a Lebesgue point and

$$
\lim_{r \to 0} m^d(B_r(x))^{-1} \int_{B_r(x)} |\nabla f(y) - \nabla f(x)|^p dm^d(y) = 0.
$$

We apply the Morrey's inequality Theorem 4.49 to

$$
v(y) = f(x+y) - f(x) - \sum_{j=1}^{d} \partial_j f(x) y_j
$$

on $B_r(0)$ where again $x \in A$. Then

$$
\begin{aligned}
|v(y)| \leq & cr^{1-\frac{d}{p}} \left(\int_{B_r(0)} |\nabla f(x+z) - \nabla f(x)|^p dm^d(z) \right)^{\frac{1}{p}} \\
= & cr \left(m^d(B_r(x))^{-1} \int_{B_r(x)} |\nabla f(z) - \nabla f(x)|^p dm^d(z) \right)^{\frac{1}{p}} \\
= & o(r).
\end{aligned}
$$

This implies that f is differentiable at $x \in A$. □

Theorem 4.51 (Sobolev)**.** *Suppose that* $1 \le p < d$ *and*

$$\frac{1}{q} + \frac{1}{d} = \frac{1}{p}.$$

Then

$$\|f\|_{L^q(\mathbb{R}^d)} \le c \||Df|\|_{L^p(\mathbb{R}^d)}$$

whenever $f \in L^q(\mathbb{R}^d)$ *and* $|Df| \in L^p(\mathbb{R}^d)$*.*

Proof. We prove the estimate first for $p = 1$ and $q = \frac{d}{d-1}$. More precisely we prove the estimate

$$\|f\|^d_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \le 2^{-d} \prod_{j=1}^{d} \|\partial_j f\|_{L^1(\mathbb{R}^d)} \tag{4.7}$$

by induction on the dimension. The case $d = 1$ has been contained in Lemma 4.48. Suppose we have proven the estimate for $d \le k-1$. Then by Fubini and Hölder's inequality (since $\frac{1}{k-1} + \frac{k-2}{k-1} = 1$)

$$\begin{aligned}
\|f\|^{\frac{k}{k-1}}_{L^{\frac{k}{k-1}}(\mathbb{R}^k)} &= \int_{\mathbb{R}} \int_{\mathbb{R}^{k-1}} |f|^{\frac{1}{k-1}} |f| dm^{k-1} dm^1(x_1) \\
&\le \int_{\mathbb{R}} \|f(x_1, \dots)\|^{\frac{1}{k-1}}_{L^1(\mathbb{R}^{k-1})} \|f(x_1, \dots)\|_{L^{\frac{k-1}{k-2}}(\mathbb{R}^{k-1})} dm^1(x_1) \\
&\le \sup_{x_1} \|f(x_1, .)\|^{\frac{1}{k-1}}_{L^1(\mathbb{R}^{k-1})} \int_{\mathbb{R}} \|f(x_1, .)\|_{L^{\frac{k-1}{k-2}}(\mathbb{R}^{k-1})} dm^1(x_1) \\
&\le 2^{-\frac{1}{k-1}} \|\partial_{x_1} f\|_{L^1(\mathbb{R}^k)} \int_{\mathbb{R}} \left(2^{-(k-1)} \prod_{j=2}^{k} \|\partial_{x_j} f(x_1, .)\|_{L^1(\mathbb{R}^{k-1})} \right)^{\frac{1}{k-1}} dm^1(x_1)
\end{aligned}$$

We take the inequality to the power $k-1$, and apply Hölder's inequality in the form

$$\left(\int \prod_{j=2}^{k} |g_j|^{\frac{1}{k-1}} dm^1 \right)^{k-1} \le \prod_{j=2}^{k} \int |g_j| dm^1$$

to arrive at

$$\|f\|^k_{L^{\frac{k}{k-1}}(\mathbb{R}^k)} \le 2^{-k} \prod_{j=1}^{k} \|\partial_j f\|_{L^1(\mathbb{R}^k)}.$$

Now let $1 < p < d$. We apply the above inequality (4.7) to $|f|^{\frac{(d-1)p}{d-p}}$. Then

$$\begin{aligned}
\|f\|^{\frac{(d-1)p}{d-p}}_{L^q(\mathbb{R}^d)} &= \||f|^{\frac{(d-1)p}{d-p}}\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \\
&\le \|D|f|^{\frac{(d-1)p}{d-p}}\|_{L^1(\mathbb{R}^d)} \le \frac{(d-1)p}{d-p} \int |f|^{\frac{d(p-1)}{d-p}} |Df| dm^d \\
&\le \frac{(d-1)p}{d-p} \|f\|^{\frac{d(p-1)}{d-p}}_{L^q(\mathbb{R}^d)} \|Df\|_{L^p(\mathbb{R}^d)}
\end{aligned}$$

where we first argue for smooth functions, and where we used Hölder's inequality in the last step. □

[11.01.2017]

[13.01.2017]

4.8 Applications to PDE

Let $U \subset \mathbb{R}^d$ be open and bounded, $f : U \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be continuous and suppose that there exists $1 \le p \le r < \infty$ such that

$$\frac{1}{r} + \frac{1}{d} \ge \frac{1}{p}$$

and

$$|f(x,u,P)| \le c(1 + |u|^r + |P|^p). \tag{4.8}$$

If $u \in W^{1,p}(U)$ then

$$x \to f(x, u(x), \nabla u(x))$$

is measurable and integrable since

$$\begin{aligned}\int_U |f(x,u(x),\nabla u(x))| dm^d &\le c(m^d(U) + \int_U |u|^r dm^d + \int_U |\nabla u|^p dm^d) \\ &\le \tilde{c}(1 + \|u\|^r_{L^r(U)} + \|u\|^p_{W^{1,p}(U)}),\end{aligned}$$

where we put all the constants into $\tilde{c}$. If $1 \le p < d$ by Hölder's, Poincaré's and Sobolev's inequality, with $\frac{1}{q} + \frac{1}{d} = \frac{1}{p}$ and

$$\alpha = \frac{\frac{1}{p} - \frac{1}{r}}{\frac{1}{p} - \frac{1}{q}},$$

we have

$$\|u\|_{L^r(U)} \le \|u\|^{1-\alpha}_{L^p(U)} \|u\|^{\alpha}_{L^q(U)} \le C\|u\|_{W^{1,p}(U)},$$

where q is the exponent of the Sobolev inequality. Here we assume that either $U \subset \mathbb{R}^d$ such that Whitney extension Theorem 4.42 holds true, or $u \in W^{1,p}_0(U)$.

Definition 4.52. *Let $U \subset \mathbb{R}^d$ be open and bounded,*

$$F^j, f \in C(U \times \mathbb{R} \times \mathbb{R}^d)$$

for $1 \le j \le d$ and suppose these functions to satisfy (4.8). *Then $u \in W^{1,p}(U)$ is called weak solution to*

$$\sum_{j=1}^d \partial_j F^j(x,u,\nabla u) = f(x,u,\nabla u)$$

if this identity holds in the sense of distributions. This holds iff

$$\int_U \sum_{j=1}^d F^j(x,u,\nabla u)\partial_j \phi + f(x,u,\nabla u)\phi \, dm^d = 0$$

for all $\phi \in \mathcal{D}(U)$.

Remark 4.53. *We may replace the condition $\phi \in \mathcal{D}(U)$ by $\phi \in C^1(\overline{U})$ with $\phi|_{\partial U} = 0$. If $1 < p$ we may replace it by $\phi \in W_0^{1,q}(U)$ where $\frac{1}{q} + \frac{1}{p} = 1$.*

We turn to a particular case. Let

$$a^{ij} \in L^\infty(U)$$

$$F^j, f \in L^p(U)$$

for $1 \leq i,j \leq d$. Then $u \in W^{1,p}(U)$ is a weak solution to

$$\sum_{i,j=1}^d \partial_i (a^{ij}(x)\partial_j u) = \sum_{j=1}^d \partial_j F^j + f$$

if and only if

$$\int_U \sum_{i,j=1}^d a^{ij}(x)\partial_j u \overline{\partial_i \phi} - \sum_{j=1}^d F^j(x)\overline{\partial_j \phi} + f\bar{\phi} \, dm^d = 0$$

for all $\phi \in C^1(\overline{U})$ with $\phi|_{\partial U} = 0$.

If $g \in W^{1,p}(U)$ we say $u = g$ on ∂U in the $W^{1,p}$ sense if

$$u - g \in W_0^{1,p}(U).$$

Definition 4.54. *We call the $(a^{ij})_{1 \leq i,j \leq d}$ elliptic if there exists $\kappa > 0$ so that*

$$\operatorname{Re} \sum_{i,j=1}^d a^{ij}(x)\xi_i \overline{\xi_j} \geq \kappa |\xi|^2$$

for almost all $x \in U$ and all $\xi \in \mathbb{C}^d$.

Let $F^j, f \in L^2(U)$, $g \in W^{1,2}(U)$. We consider the boundary value problem

$$\sum_{i,j=1}^d \partial_i(a^{ij}\partial_j u) = \sum_{j=1}^d \partial_j F^j + f \qquad \text{in } U$$

$$u = g \qquad \text{on } \partial U$$

Theorem 4.55. *There exists exactly one weak solution $u \in W^{1,2}(U)$ which satisfies the boundary condition in the $W^{1,2}$ sense. The map*

$$(L^2(U))^{d+1} \times W^{1,2}(U) \ni (F, f, g) \to u \in W^{1,2}(U)$$

is a continuous linear map.

Proof. **Step 1: Reduction** Formally setting $w = u - g$ it suffices to find $w \in W_0^{1,2}(U)$ such that

$$\sum_{i,j=1}^{d} \partial_i(a^{ij}\partial_j w) = \sum_{j=1}^{d} \partial_j(F^j - \sum_{i=1}^{d} a^{ji}\partial_i g) + f \qquad \text{in } U$$

which reduces the problem to the case $g = 0$.

Step 2: The Hilbert space We recall that $W_0^{1,2}(U)$ is a Hilbert space - we can consider it as a closed subspace of $(L^2(U))^{d+1}$, and closed subspaces of Hilbert spaces are Hilbert spaces. The inner product is

$$\langle u, v\rangle = \int_U \sum_{j=1}^{d} \partial_j u \overline{\partial_j v} dm^d$$

Step 3: The quadratic form We define the quadratic form

$$A(u, v) = \int_U \sum_{i,j=1}^{d} a^{ij}\partial_j u \overline{\partial_i v} dm^d.$$

The quadratic form is continuous - there exists $C > 0$ so that

$$|A(u, v)| \le C\|u\|_{W^{1,2}(U)}\|v\|_{W^{1,2}(U)}$$

and it satisfies

$$\operatorname{Re} A(u, u) \ge \kappa \int_U |\nabla u|^2 dm^d$$

for all $u, v \in W^{1,2}(U)$. By the Poincaré inequality in $W_0^{1,2}(U)$ (since U is bounded) there exists $c > 0$ so that

$$\|u\|_{L^2(U)} \le c\|\nabla u\|_{L^2}$$

and thus there exists $\tilde{\kappa} > 0$ so that

$$\operatorname{Re} A(u, u) \ge \tilde{\kappa}\|u\|^2_{W_0^{1,2}(U)}.$$

Step 4: The linear form Let F and f be as in the theorem. Then

$$v \to Lv = \int_U -\sum_{j=1}^{d} F^j \partial_j v + fv$$

satisfies

$$\|L\|_{(W_0^{1,2}(U))^*} \le \|F\|_{(L^2(U))^d} + \|f\|_{L^2(U)}$$

Step 5: The lemma of Lax-Milgram. By the Lemma of Lax-Milgram there is a unique $u \in W_0^{1,2}(U)$ such that

$$A(u,v) = L(\overline{v}), \quad \forall v \in W_0^{1,2}(U),$$

which is then a weak solution. The map $L \to u$ is linear and continuous.

Step 6: The bound Let $v = u$ and take the real part. Then

$$\begin{aligned}\kappa\|\nabla u\|_{L^2}^2 \le& \|F\|_{L^2}\|\nabla u\|_{L^2} + \|f\|_{L^2}\|u\|_{L^2}\\ \le& \|F\|_{L^2}\|\nabla u\|_{L^2} + C\|f\|_{L^2}\|\nabla u\|_{L^2}\end{aligned}$$

and hence

$$\kappa\|\nabla u\|_{L^2} \le \|F\|_{L^2} + C\|f\|_{L^2}.$$

we complete the bound by a second application of the Poincaré inequality.

□

There is a particular case $a^{ij} = \delta^{ij}$ and $F = 0$. It becomes the Poisson problem

$$-\Delta u = f \qquad \text{in } U$$

$$u = g \qquad \text{on } \partial U$$

We obtain a unique weak solution for U open and bounded, $f \in L^2(U)$ and $g \in W^{1,2}(U)$.

If there is a barrior at every boundary points then the problem

$$-\Delta u = 0 \qquad \text{in } U$$

$$u = g \qquad \text{on } \partial U$$

has a unique solution $u \in C^1(U) \cap C(\overline{U})$ for every continuous g. We call this solution classical.

A classical solution is not necessarily in $W^{1,2}$, and a weak solution is not necessarily in $C(\overline{U})$. In most important cases both solutions are identical.

[13.01.2017]

[18.01.2017]

5 Linear Functionals

In this section we will study the dual space X^* of Banach spaces X.

5.1 The Theorem of Hahn-Banach

Definition 5.1. *Let X be a $\mathbb{K}$ vector space. A map $p : X \to \mathbb{R}$ is called sublinear if*

1. *$p(\lambda x) = \lambda p(x)$ for $x \in X$ and $\lambda \geq 0$,*
2. *$p(x+y) \leq p(x) + p(y)$ for $x, y \in X$.*

Examples:

1. The norm of a normed space is sublinear.
2. If $\mathbb{K} = \mathbb{R}$, any element of X^* is sublinear.
3. The Minkowski functional of a convex set. Let $K \subset X$ be convex such that for every $x \in X$ there exists $\lambda > 0$ so that $\lambda x \in X$. we define

$$p_K(x) = \inf\{\lambda > 0 : \frac{1}{\lambda} x \in K\} \in [0, \infty).$$

   It is not difficult to verify that p_K is sublinear. A norm is the Minkowski functional of the unit ball.

Theorem 5.2 (Hahn-Banach, real case)**.** *Let X be a real vector space, $Y \subset X$ a subvector space, $p : X \to \mathbb{R}$ sublinear and $l : Y \to \mathbb{R}$ linear such that*

$$l(y) \leq p(y) \qquad \text{for all } y \in Y.$$

Then there exists $L : X \to \mathbb{R}$ linear so that

1. *$l(y) = L(y)$ for all $y \in Y$*
2. *$l(x) \leq p(x)$ for all $x \in X$.*

Proof. There are two very different steps.

Suppose that $Y \neq X$. Then there exists $x_0 \in X \backslash Y$. Let Y_1 be the space spanned by Y and x_0. Every element of Y_1 can uniquely be written as

$$y + r x_0, \quad y \in Y, r \in \mathbb{R}.$$

We want to find a linear map $l_1 : Y_1 \to \mathbb{R}$ such that

1. $l_1(y) = l(y)$ for $y \in Y$
2. $l_1(y + s x_0) \leq p(y + s x_0)$ for $s \in \mathbb{R}$ and $y \in Y$.

By the first condition and linearity we have to find $t = l_1(x_0)$ so that

$$l(y) + st \leq p(y + sx_0)$$

for all $y \in Y$ and $s \in \mathbb{R}$. We consider $s > 0$. Then this inequality is equivalent to

$$st \leq s(p(y/s + x_0) - l(y/s))$$

for all $s > 0$ and $y \in Y$

$$\iff t \leq \inf_y p(y + x_0) - l(y).$$

Similarly the inequality holds for $s < 0$ if and only if

$$t \geq \sup_y l(y) - p(y - x_0).$$

We can find t if and only if

$$l(y) - p(y - x_0) \leq p(\tilde{y} + x_0) - l(\tilde{y}) \qquad \text{for all } y, \tilde{y} \in Y$$

which follows from the inequality on Y and sublinearity of p:

$$l(y) + l(\tilde{y}) = l(y + \tilde{y}) \leq p(y + \tilde{y}) \leq p(y + x_0) + p(\tilde{y} - x_0).$$

This completes the first step.

For the second step we need the axiom of choice in the form of Zorn's lemma.

Let Z be a partially ordered set which contains an upper bound for every chain. Then there is a maximal element.

A chain is a totally ordered subset, i.e. a subset A so that always either $a \leq b$ or $b \leq a$. An element b is an upper bound for the chain A, if $a \leq b$ for all $a \in A$. An element $a \in Z$ is maximal if $b \in Z$, $b \geq a$ implies $b = a$.

We define

$$Z = \{(W, l_W) : Y \subset W,\, l_W|_Y = l,\, l_W(w) \leq p(w) \text{ for } w \in W\},$$

with the ordering

$$(W, l_W) \leq (V, l_V) \qquad \text{if } W \subset V \quad \text{and } l_V|_W = l_W.$$

This is a partial order. If $\tilde{Z}$ is a chain then

$$V = \bigcup_{(W, l_W) \in \tilde{Z}} W$$

with the obvious l_V being an upper bound for the chain. Now let (V, l_V) be a maximal element. If $V = X$ we are done. Otherwise we obtain a contradiction by the first step. $\square$

There is a complex version. It relies on the observation

Lemma 5.3. *Let X be a complex vector space and $l : X \to \mathbb{R}$ be $\mathbb{R}$ linear. Then it is the real part of the linear map*

$$l_{\mathbb{C}}(x) = l(x) - il(ix).$$

The real part determines $l_{\mathbb{C}}$.

Proof. We have to show complex linearity. Real linearity is obvious. We compute

$$l_{\mathbb{C}}(ix) = l(ix) - il(iix) = i(l(x) - il(ix)) = il_{\mathbb{C}}(x).$$

□

Theorem 5.4 (Complex Hahn Banach)**.** *Let X be a complex vector space, Y a subvector space, p sublinear and $l : Y \to \mathbb{C}$ linear so that*

$$\operatorname{Re} l(y) \le p(y) \text{ for } y \in Y.$$

Then there exists $L : X \to \mathbb{C}$ linear so that

1. $L|_Y = l$
2. $\operatorname{Re} L(x) \le p(x)$.

Proof. We apply the real theorem of Hahn Banach to the real part, and extend it to a complex linear map by Lemma 5.3. To complete the proof we observe that L and $\tilde{L}$ are the same iff the real parts are the same. □

We formulate the consequences for normed vector spaces, making use of the fact that norms are sublinear.

Theorem 5.5. *Let X be a normed $\mathbb{K}$ vector space, Y a subspace and $l : Y \to \mathbb{K}$ a continuous linear map. Then there exists $L : X \to \mathbb{K}$ linear and continuous so that*

1. $L|_Y = l$
2. $\|L\|_{X^*} = \|l\|_{Y^*}$.

Proof. We define

$$p(x) = \|l\|_{Y^*}\|x\|_X.$$

Then

$$\operatorname{Re} l(y) \le p(y)$$

for all $y \in Y$. We apply the theorems of Hahn-Banach to obtain $L \in X^*$ so that $L|_Y = l$ and

$$\operatorname{Re} L(x) \le p(x).$$

This implies

$$|L(x)| = \operatorname{Re} \alpha L(x) = \operatorname{Re} L(\alpha x) \leq p(\alpha x) = p(x) = \|l\|_{Y^*} \|x\|_X$$

for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. Thus

$$\|L\|_{X^*} = \|l\|_{Y^*}.$$

□

5.2 Consequences of the theorems of Hahn-Banach

Lemma 5.6. *Let X be a normed space and $x \in X$. There exists $x^* \in X^*$ with $\|x^*\|_{X^*} = 1$ and $x^*(x) = \|x\|_X$.*

Proof. Let $x \in X \backslash \{0\}$ and let Y be the span of x. Y is one dimensional and we define $y^* \in Y^*$ by $y^*(rx) = r\|x\|_X$. Then $\|y^*\|_{Y^*} = 1$. We apply Theorem 5.5 to obtain x^*. If $x = 0$ we choose $x_0 \neq 0$ and find x^* for x_0. □

Corollary 5.7. *Let X be a normed space. If $x \in X$ then*

$$\|x\|_X = \sup\{\operatorname{Re} x^*(x) : x^* \in X^*, \|x^*\|_{X^*} = 1\}.$$

If $x^* \in X^*$ then

$$\|x^*\|_{X^*} = \sup\{\operatorname{Re} x^*(x) : x \in X, \|x\|_X = 1\}.$$

Proof. The first claim is a consequence of Lemma 5.6. The second statement is an immediate consequence of the definition. □

Lemma 5.8. *If X is a normed space and $Y \subset X$ is a closed subspace, $Y \neq X$, then there exists $x^* \in X^*$, $x^* \neq 0$ with $x^*|_Y = 0$.*

Proof. Let $x_0 \notin Y$. We define

$$l(y + tx_0) = t$$

and extend it to X by Theorem 5.5. □

Corollary 5.9. *Let X^{**} be the dual space of X^*. There is a cannonical isometry*

$$J : X \to X^{**}, \quad J(x)(x^*) = x^*(x).$$

Proof. Only the isometry has to be shown

$$\|J(x)\|_{X^{**}} = \sup_{\|x^*\|_{X^*}=1} \operatorname{Re} J(x)(x^*) = \sup_{\|x^*\|_{X^*}=1} x^*(x) = \|x\|_X.$$

□

Remark 5.10. *If $X = L^p(\mu)$, $1 < p < \infty$ then X^* is isomorphic to $L^{\frac{p}{p-1}}(\mu)$ and J is surjective.*

Lemma 5.11. $(l^\infty)^* \neq l^1$.

Proof. The space of converging sequences c is a closed subspace of l^∞. Let $l : c \to \mathbb{K}$ be defined by

$$l((x_j)) = \lim_{j\to\infty} x_j.$$

Then

$$|l((x_j))| \le \|(x_j)\|_{l^\infty}$$

for every converging sequence. By Theorem 5.5 it has an extension L to l^∞. Clearly $L(e_j) = l(e_j) = 0$. We claim that it cannot be represented in the form

$$L((x_j)) = \sum_{j=1}^\infty y_j x_j$$

for $(y_j) \in l^1$ - if it were represented in this fashion then all y_j would have to vanish. $\square$

In particular $J : l^1 \to (l^1)^{**}$ is not surjective.

[18.01.2017]

[20.01.2017]

Lemma 5.12. *If X is a normed space and X^* is separable then X is separable.*

Proof. Since X^* is separable, and a subset of a separable set is separable also the unit sphere is separable. Let x_j^* be a dense sequence of unit vectors. We choose a sequence of unit vectors x_j with $x_j^*(x_j) \ge \frac{1}{2}$. We claim that the span U of the x_j is dense. Otherwise, by Lemma 5.8 there exists $x^* \in X^*$ of norm 1 which vanishes on the closure of U, and in particular $x^*(x_j) = 0$ for all j. By density there exists j such that $\|x^* - x_j^*\|_{X^*} < \frac{1}{2}$ and hence

$$\frac{1}{2} \le \operatorname{Re} x_j^*(x_j) = \operatorname{Re}(x^*(x_j) + (x_j^* - x^*)(x_j)) < \frac{1}{2}$$

This is a contradiction. $\square$

Lemma 5.13. *Let $U \subset \mathbb{R}^d$ be open, $1 \le p < \infty$ and $k \in \mathbb{N}$. The map*

$$J : L^{\frac{p}{p-1}}(U \times \Sigma_k) \ni (g_\alpha) \to (f \to \sum_{|\alpha|\le k} g_\alpha \partial^\alpha f) \in (W^{k,p}(U))^*$$

is bounded and surjective.

Proof. The map

$$W^{k,p}(U) \ni f \to (\partial^\alpha f)_{|\alpha|\le k} \in L^p(U \times \Sigma_k)$$

map $W^{k,p}$ isometrically to a closed subspace. Any $y^* \in (W^{k,p})^*$ defines a linear functional on this closed subspace. By Theorem 5.5 we can extend it to the whole of $L^p(U \times \Sigma_k)$. This can be represented by a function in $L^{\frac{p}{p-1}}(U \times \Sigma_k)$. We have seen that J has norm 1 in an exercise. $\square$

5.3 Separation theorems

Lemma 5.14. *Let X be a normed space and $K \subset X$ convex. If 0 is in the interior of K then for every $x \in X$ there exists $\lambda > 0$ so that $\lambda x \in K$. Moreover there exists $C > 0$ so that*

$$p_K(x) \le C\|x\|_X.$$

If X is a Banach space, and for every x there exists $\lambda > 0$ so that $\lambda x \in K$ then 0 is in the interior of K.

Proof. If 0 is in the interior of K there exists $\varepsilon > 0$ so that $B_\varepsilon(0) \subset K$. An easy calculation shows that then $p_K(x) \le \varepsilon^{-1}\|x\|_X$.

Suppose that X is Banach and that for every x there exists $\lambda > 0$ so that $\lambda x \in K$. In particular $0 \in K$. Let

$$A_n = \{x \in X : \frac{1}{n}x \in \overline{K}\}.$$

The set A_n are closed, convex, $0 \in A_n$ and $X = \bigcup A_n$. By the Baire category theorem one and hence all of the A_n have nonempty interior. In particular there exist x and $\varepsilon > 0$ so that $B_\varepsilon(x) \subset A_1$. There exists A_n so that $-x \in A_n$ hence $-x/n \in A_1$. The convex hull of $B_\varepsilon(x)$ and $-x/n$ contains a ball around 0. Hence 0 is in the interior of $A_{\frac{1}{2}} \subset \overline{K}$. $\square$

Lemma 5.15. *Let X be a normed vector space and V convex, open with $0 \notin V$. Then there exists $x^* \in X^*$ with*

$$\operatorname{Re} x^*(x) < 0 \qquad \text{if } x \in V$$

Proof. It suffices to consider $\mathbb{K} = \mathbb{R}$. The complex case is a consequence of Lemma 5.3. Let $x_0 \in V$ and define the translate $U = V - x_0$. Let p_U be the Minkowski functional of U. It is sublinear. Let $y_0 = -x_0 \notin U$ and Y the span of y_0. We define

$$l(ty_0) = tp_U(y_0) \qquad t \in \mathbb{R}.$$

Then $l(y) \le p(y)$ for all $y \in Y$. By Theorem 5.2 there exists $L \in X^*$ with $L|_Y = l$, $L(x) \le p(x)$ for $x \in X$ (here we use Lemma 5.14). In particular $L(y_0) \ge 1$ and for $x \in V$ and $u = x + y_0$

$$L(x) = L(u) - L(y_0) \le p_U(u) - 1 < 0.$$

□

Theorem 5.16 (Separation theorem 1)**.** *Let X be a normed space, V and W disjoint convex sets with V open. Then there exists $x^* \in X^*$ such that*

$$\operatorname{Re} x^*(v) < \operatorname{Re} x^*(w) \qquad \textit{for every } v \in V, w \in W$$

Proof. Let $U = V - W$. It is convex and open. Since V and W are disjoint $0 \notin U$. By Lemma 5.15 there exist $x^* \in X^*$ so that $\operatorname{Re} x^*(x) < 0$ for $x \in U$. This implies the desired inequality. □

Theorem 5.17 (Separation theorem 2)**.** *Let X be a normed space, V convex and closed, $x \notin V$. Then there exists $x^* \in X^*$ such that*

$$x^*(x) < \inf_{v \in V} x^*(v). \tag{5.1}$$

Proof. We may assume $x = 0$. Since V is closed there exists $\varepsilon > 0$ so that $B_\varepsilon(0) \cap V = \{\}$. We apply Theorem 5.16 to see that there is $x^* \in X^*$ with

$$\operatorname{Re} x^*(u) < \operatorname{Re} x^*(v) \qquad \text{for } u \in B_\varepsilon(0), v \in V$$

There exists $x \in B_\varepsilon(0)$ so that

$$\operatorname{Re} x^*(x) \ge \varepsilon \|x^*\|_{X^*}/2 > 0 = x^*(0)$$

which implies (5.1) . □

Corollary 5.18. *Let X be a normed vector space, $K \subset X$ comvex and $x \in \partial K$. Then there exists a half space containing K with x a boundary point.*

Proof. Exercise □

5.4 Weak* topology and the theorem of Banach-Alaoglu

Let X be a normed space. The dual space X^* is a Banach space, and hence a metric space. The metric defines open sets, and hence a topology which we call norm topology.

Definition 5.19. *Let A be a set. A family τ of sets is called topology if*

1. *$\{\}, A \in \tau$.*

2. *$B, C \in \tau$ implies $B \cap C \in \tau$.*

3. *If Λ is a set, and for every $\lambda \in \Lambda$ there is a set $B_\lambda \in \tau$ then $\bigcup_{\lambda \in \Lambda} B_\lambda \in \tau$.*

We call the elements of τ open. A map is called continuous if the preimage of open sets is open. A set is called compact, if it is Borel and every open covering contains a finite subcover.

We want to define a topology on X^*. Desired properties are

1. For every x the map $X^* \ni x^* \to x^*(x) \in \mathbb{K}$ is continuous. Equivalently, for every open set $U \in \mathbb{K}$ and $x \in X$

$$U_x^* = \{x^* : x^*(x) \in U\}$$

is open.

2. The weak$*$ topology is the weakest topology with this property. This means that the open sets are the smallest subset of the power set, such that all sets above are contained in it, and arbitrary unions and finite intersections are contained in it.

Finite intersections of sets of the first type are sets

$$\bigcap_{j=1}^{N} (U_j)_{x_j}^* \tag{5.2}$$

and open sets are arbitrary unions of such sets. This follows from a multiple application of the distributive law of union and intersection.

Definition 5.20. *A local base $\mathcal{S}$ of a topology is a family of open sets so that for every x and every open set U there exists $V \in \mathcal{S}$ so that $x \in V \subset U$. A subbase is a collection of open sets so that finite intersections form a local base.*

Examples.

1. In metric space the balls $\{B_{1/n}(x) : x \in X, n \in \mathbb{N}\}$ are a base.

2. The sets (5.2) form a base.

3. The sets $\{U_x^* : x \in X, U \subset \mathbb{K} \text{ open }\}$ are a subbase.

Lemma 5.21 (Alexander)**.** *If $\mathcal{S}$ is a subbase of a topology, then X is compact if every $\mathcal{S}$ cover has a finite subcover.*

Proof. We argue by contrapositive and assume that X is not compact. Let $\mathcal{P}$ be the collection of all covers without finite subcover. By assumption $\mathcal{P}$ is not empty. We take the partial order by inclusion. The union of every element of a chain is an upper bound. By Zorn's lemma there is a maximal cover Γ without finite subcover.

Now let $\tilde{\Gamma} = \Gamma \cap \mathcal{S}$. It has no finite subcover since it is a subset of Γ. We show that $\tilde{\Gamma}$ covers X, which gives the conclusion.

Arguing by contradiction assume that $x \in X$ is in none of the elements of $\tilde{\Gamma}$. Γ covers X hence there is $W \in \Gamma$ so that $x \in W$. Since $\mathcal{S}$ is a subbase there are $V_j \in \mathcal{S}$ so that $\cap_{j=1}^N V_j \subset W$. Since x is not covered by $\tilde{\Gamma}$, $V_j \notin \Gamma$.By maximality for each j $\Gamma \cup \{V_j\}$ has a finite subcover

$$X = \bigcup_{j=1}^N \bigcup_{k=1}^{M_j} Y_{jk} \cup V_j$$

Hence

$$X = W \cup \bigcup_{j=1}^N \bigcup_{k=1}^{M_j} Y_{jk}$$

is a finite subcover of Γ which is a contradiction. □

[20.01.2017]

[25.01.2017]

Let D be a set, suppose that for every $\alpha \in D$ there is a set X_α. The cartesian product

$$X = \prod_{\alpha \in D} X_\alpha$$

is the set of all 'maps' which assign to each α the element of X_α. There are the obvious projections

$$\pi_\alpha : X \to X_\alpha$$

Suppose that all spaces X_α are topological spaces. Let τ be the smallest topology (subset of the power set) containing all preimages of open sets in X_α under π_α.

Lemma 5.22. *The preimages of open sets in X_α under π_α define a subbase $\mathcal{S}$ of τ.*

Proof. We define the collection of arbitrary unions of finite intersections of such sets. Then arbitrary unions and finite intersections have this form. Thus every open set in τ is a union of finite intersections of such sets. Thus these sets are a subbase. □

Theorem 5.23 (Tychonoff)**.** *Any cartesian product X of compact sets X_α with topology τ is compact.*

Proof. Let Γ be a $\mathcal{S}$ cover. Let Γ_α consists of the sets whose preimages under π_α are in Γ. Assume that no Γ_α covers X_α. Then for all α there exists $x_\alpha \in X_\alpha$ so that x_α is not covered by Γ_α . Then $x \in X$ with $\pi_\alpha(x) = x_\alpha$ is not covered by Γ . This is a contradiction. Hence at least one Γ_α covers X_α. Since X_α is compact, a finite subset of Γ_α covers X_α and hence the preimages of this finite subset under π_α covers X. By Alexander's theorem X is compact. □

Theorem 5.24 (Banach-Alaoglu)**.** *Let X be a normed space. The closed unit ball $\Lambda \subset X^*$ is compact in the weak* topology.*

Proof. Denote $\overline{B_1^X(0)} = \{x \in X \,|\, \|x\|_X \le 1\}$ and $\overline{B_1(0)} \subset \mathbb{K}$. Then

$$Z = \{f : \overline{B_1^X(0)} \to \overline{B_1(0)}\}$$

is the Cartesian product of compact sets. We consider Λ as subset of Z. Then Λ carries two topologies: The weak topology, and the topology as subset of Z. We claim that the two are the same. But this is an immediate consequence of the definitions of the topologies.

Now let f_0 be in the closure of Λ (closed sets are complements of open sets, and the closure is the smallest closed set containing Λ). We claim that f_0 is linear. Let $\|x\|_X, \|y\|_X \le 1$ such that $\|\alpha x + \beta y\|_X \le 1$ and $\varepsilon > 0$. The set of all $f \in \Lambda$ with

$$\max\{|f(x) - f_0(x)|, |f(y) - f_0(y)|, |f(\alpha x + \beta y) - f_0(\alpha x + \beta y)|\} < \varepsilon$$

is open and contains a linear f of norm 1. Hence

$$|f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)| < \varepsilon + |\alpha|\varepsilon + |\beta|\varepsilon$$

for all $\varepsilon > 0$. Hence

$$f_0(\alpha x + \beta y) = \alpha f_0(x) + \beta f_0(y)$$

for all such x, y, α, β and thus f_0 is linear. Similarly the norm of $f_0 \le 1$. Thus Λ is closed in the compact set Z. Any closed subset of a compact set is compact. □

Definition 5.25. *Let X be a Banach space and X^* its dual. We say a sequence $x_n^* \in X^*$ is weak* convergent to x^* if*

$$x_n^*(x) \to x^*(x)$$

for all $x \in X$. We write

$$x_n^* \rightharpoonup^* x^*.$$

We say $x_n \in X$ converges weakly to $x \in X$ if

$$x^*(x_n) \to x^*(x)$$

for all $x^* \in X^*$. We then write

$$x_n \rightharpoonup x.$$

Lemma 5.26. *Weak and weak* convergent sequences are bounded.*

Proof. The statement follows for the weak* converging subsequences by the uniform boundedness principle. We may consider weakly converging sequences as weak$*$ converging sequences in X^{**}, hence they are bounded. $\square$

Theorem 5.27. *Let X be a Banach space and $K \in X$ a closed bounded convex set. Let (x_j) be a weakly convergent sequence in K with weak limit x. Then $x \in K$.*

Proof. Let $x_0 \notin K$. By Theorem 5.17 there exists $x^* \in X^*$ such that

$$x^*(x_0) < \inf_{x \in K} x^*(x).$$

Thus x_0 cannot be a weak limit. $\square$

Thus

$$x_n^* \rightharpoonup^* x^* \qquad \Longrightarrow \qquad \|x^*\|_{X^*} \le \liminf_{n\to\infty} \|x_n^*\|_{X^*} \tag{5.3}$$

and

$$x_n \rightharpoonup x \qquad \Longrightarrow \qquad \|x\|_X \le \liminf_{n\to\infty} \|x_n\|_X. \tag{5.4}$$

Again (5.4) follows form (5.3) by the cannonical embedding $J : X \to X^{**}$. Inequality (5.3) follows from

$$\|x^*\|_{X^*} \le \sup \|x_n^*\|_{X^*},$$

which is a consequence of Theorem 5.27 applied with K a closed ball.

Lemma 5.28. *Suppose that X is separable. Then there exists a metric on the closed unit ball of X^* so that the topology as metric spaces is the weak$*$ topology.*

Proof. Let x_j be a dense sequence in $B_1^X(0)$. We define

$$d(x^*, y^*) = \max_j 2^{-j} \min\{1, |x^*(x_j) - y^*(x_j)|\}.$$

This is a metric. Convergence of bounded sequences in this metric is the same as weak convergence. Moreover open balls are open in the weak topology, and they form a base of the weak topology. $\square$

Theorem 5.29. *[Weak* compactness of bounded sequences] Every bounded sequence $x_j^* \in X^*$ where X is separable contains a weakly convergent subsequence.*

We will provide two proofs of this important fact.

Proof 1. Banach Alaoglu implies that the closed balls are weak* compact. By Lemma 5.28 the weak* topology comes from a metric which makes X^* with the weak* topology a metric space. For metric spaces the different notions of compactness are equivalent. $\square$

Proof 2. Let (x_l) be a dense sequence on the unit ball of X. Let (x_j^*) be a bounded sequence of element of X^*. Then there is a subsequence such that

$$(x_{j_k}^*(x_1))$$

converges to y_1 such that

$$|x_{j_k}^*(x_1) - y_1| < 2^{-k}.$$

Taking repeated subsequences we find a subsequence and a sequence y_l such that

$$|x_{\tilde{j}_k}^*(x_l) - y_l| < 2^{-k}$$

for $k \geq l$. Since x_j^* is bounded the y_l define a unique continuous linear map $x^* : X \to \mathbb{K}$. Moreover $x_j^* \rightharpoonup^* x^*$. $\square$

[25.01.2017]

[27.01.2017]

Definition 5.30. *We call a Banach space X reflexive, if $J : X \to X^{**}$ is surjective.*

Hilbert spaces are reflexive. If $1 < p < \infty$, $L^p(\mu)$ is reflexive as a consequence of Theorem 3.19.

We define the weak topology in the same fashion as the weak* topology.

Lemma 5.31. *Suppose that X is reflexive. Then the weak toplogy of X is the same as the weak* topology of X^{**}.*

Proof. Both are the smallest topology such that $x \to x^*(x) = J(x)(x^*)$ are continuous for all $x^* \in X^*$. $\square$

Corollary 5.32. *Let (X, d) be a σ compact metric space, μ a Radon measure on X and $1 < p < \infty$. Then every bounded sequence in $L^p(\mu)$ has a weakly convergent subsequence. If μ is σ-finite any bounded sequence in $L^\infty(\mu)$ has a weak* convergent subsequence.*

Proof. By Corollary 3.37 $L^p(\mu)$ is separable if $1 \leq p < \infty$. By Theorem 3.19 and Corollary 3.20 $L^{\frac{p}{p-1}}(\mu)$ is isomorphic to the dual space of $L^p(\mu)$ and vice versa when $1 < p < \infty$. By Lemma 5.31 the weak and the weak* topology are the same when $1 < p < \infty$. By Theorem 5.29 every bounded sequence has a weak* convergent subsquence. $\square$

Lemma 5.33. *Suppose the closed unit ball of a Banach space X is compact. Then its dimension is finite.*

Proof. Let X be not of finite dimension. We claim that there is a sequence of unit vectors of distance $> \frac{1}{2}$. We construct them recursively. Let $(x_n)_{n\le N}$ be chosen and let X_N denote their span. It is a closed subspace and by Theorem 5.5 there is $x^* \in X^*$ of norm 1 so that $x^*|_{X_N} = 0$. Thus there exists x_{N+1} of norm 1 so that $\operatorname{Re} x^*(x_{N+1}) > \frac{3}{4}$. Then

$$\frac{3}{4} \le \operatorname{Re} x^*(x_{N+1} - x_n) \le \|x_{N+1} - x_n\|$$

for $n \le N$. This yields the result. $\square$

We collect a number of examples and remarks.

1. A closed subspace Y of a reflexive space X is reflexive. This is seen as follows: Let $0 \ne y^{**} \in Y^{**}$. It defines an element $x^{**} \in X^{**}$ by $x^{**}(x^*) = y^{**}(x^*|_Y)$. Since X is reflexive there exists $x \in X$ with $x^{**}(x^*) = x^*(x)$. We claim that $x \in Y$. If not there exists $x^* \in X^*$ such that $x^*(x) = 1$ and $x^*|_Y = 0$. This would imply

$$0 = y^{**}(x^*|_Y) = x^{**}(x^*) = x^*(x) = 1$$

which is a contradiction, and hence $y := x \in Y$. If $y^* \in Y^*$ and $x^* \in X^*$ satisfies $x^*|_Y = y^*$ then

$$y^*(y) = x^*(x) = x^{**}(x^*) = y^{**}(y^*).$$

Thus $J : Y \to Y^{**}$ is surjective.

2. Let $1 < p < \infty$, $U \subset \mathbb{R}^d$, $u_n, u \in L^p(U)$. Then

$$u_n \rightharpoonup u \quad \Longleftrightarrow \quad \sup_n \|u_n\|_{L^p} < \infty \text{ and } u_m \to u \quad \text{as distributions.}$$

To see this, assume that $u_n \rightharpoonup u$. Then $\sup_n \|u_n\|_{L^p} < \infty$ and for all $\phi \in \mathcal{D}(U) \subset L^{\frac{p}{p-1}}(U)$

$$\int u_n \phi dx \to \int u \phi dx.$$

Vice versa: Suppose that $\sup_n \|u_n\|_{L^p} < \infty$. The same is true for every subsequence. Then every subsequence (u_{n_j}) has a weakly converging subsequence, $u_{n_{j_k}} \rightharpoonup \tilde u$. Then for all $\phi \in \mathcal{D}(U)$

$$\int (\tilde u - u)\phi dx = 0$$

and thus $\tilde u = u$. This is independent of the subsequence, and hence $u_n \rightharpoonup u$. A similar statement holds for $L^\infty(U)$ and weak* convergence.

3. Let $\phi \in C_0^\infty(\mathbb{R})$ with $\int \phi dx = 1$. We define

$$u_n(x) = \phi(x - n).$$

Then $u_n(x) \to 0$ for all x and $\|u_n\|_{L^p} = \|u_0\|_{L^p(\mathbb{R})} \neq 0$ for $1 \leq p \leq \infty$. Clearly, for all $\psi \in \mathcal{D}(\mathbb{R})$, $\int u_n \psi dx \to 0$ as $n \to \infty$,

$$u_m \to 0 \qquad \text{as distributions,}$$

hence $u_n \rightharpoonup 0$ in L^p for $1 < p < \infty$. However u_n does not converge in $L^p(\mathbb{R})$ since the norm limit is also the weak limit if the first exists, but 0 cannot be the norm limit since the norm of u_n does not depend on n.

4. Let $u_n(x) = e^{inx}\chi_{[0,1]}$. Then $\|u_n\|_{L^p} = 1$. If $\phi \in \mathcal{D}(\mathbb{R})$ then

$$\begin{aligned}\int u_n \phi \, dx &= \int_0^1 e^{inx} \phi dx \\ &= -\frac{1}{in} \int_0^1 e^{inx} \phi' dx + \frac{1}{in}(e^{in}\phi(1) - \phi(0)) \\ &\to 0 \qquad \text{as } n \to \infty.\end{aligned}$$

Thus $u_n \to 0$ as distribution and as above $u_n \rightharpoonup 0$ as $n \to \infty$.

5. Let $1 < p < \infty$ and

$$u_n(x) = n^{\frac{1}{p}} e^{-(nx)} \chi_{(0,\infty)}.$$

Then

$$\int |u_n|^p dx = n \int_0^\infty e^{-p(nx)} dx = \int_0^\infty e^{-px} dx = \|u_1\|_{L^p}^p.$$

It is not hard to see that for all $\phi \in \mathcal{D}(\mathbb{R})$

$$\int u_n \phi dx = n^{-\frac{p-1}{p}} (\phi(0) + O(n^{-1})) \to 0 \qquad \text{as } n \to \infty.$$

Thus again $u_n \rightharpoonup 0$ as $n \to \infty$.

6. Let

$$u_n = n^{-\frac{1}{p}} e^{-(x/n)} \chi_{(0,\infty)}.$$

Again $\|u_n\|_{L^p} = \|u_1\|_{L^p}$,

$$\|u_n\|_{sup} = n^{-\frac{1}{p}} \to 0$$

and as above $u_n \rightharpoonup 0$ as $n \to \infty$.

5.5 The direct method in the calculus of variations

To motivate the study below we consider a specific problem. Let $U \subset \mathbb{R}^d$ be open and bounded, $f \in L^2(U)$, $g \in W^{1,2}(U)$. We define

$$E(u) = \int_U \frac{1}{2}|Du|^2 + fudx$$

for $u \in W^{1,2}(U)$. Let

$$\mathcal{A} = \{u \in W^{1,2}(U) : u - g \in W_0^{1,2}(U)\}.$$

Suppose that $v \in \mathcal{A}$ minimizes $E(u)$ in $\mathcal{A}$. If $w \in W_0^{1,2}(U)$ and $t \in \mathbb{R}$ then

$$\begin{aligned} E(v) \leq & E(v + tw) \\ = & \int_U \frac{1}{2}|D(v + tw)|^2 + f(v + tw)dx \\ = & E(v) + t\int_U \sum_{j=1}^d \partial_{x_j} v \partial_{x_j} w + fwdx + \frac{1}{2}t^2 \int_U |Dw|^2 dx. \end{aligned}$$

This is true for all t hence

$$\int_U \sum_{j=1}^d \partial_{x_j} v \partial_{x_j} w + fwdx = 0$$

for $w \in \mathcal{D}(U) \subset W_0^{1,2}(U)$. Thus

$$\Delta v = f$$

in the sense of distributions, and $v = g$ at ∂U in the Sobolev sense.

We attempt to find solutions to such equations by minimzing an energy like E.

Lemma 5.34. *Let $U \subset \mathbb{R}^d$ be open and bounded, $1 < p < \infty$ and either $X = W_0^{1,p}(U)$ or $X = W^{1,p}(U)$ under the assumption that the Whitney extension property holds. Let $(u_n) \in X$ be a bounded sequence. Then there is a subsequence (u_n) and $u \in X$ satisfying*

1. *u_n converges weakly to u in $W^{1,p}(U)$.*
2. *$\partial_{x_k} u_n$ converges weakly to $\partial_{x_k} u$.*
3. *u_n converges to u in $L^p(U)$.*

Proof. We consider $W^{1,p}(U)$ resp. $W_0^{1,p}(U)$ as closed subspace of $L^p(U \times \Sigma)$. Let (u_n) be a bounded sequence. Then there is a weakly convergent subsequence by Corollary 5.32 with weak limit u. A linear space is convex and by Theorem 5.27 we can consider u as element of X.

In both cases there is an extension to $W^{1,p}(\mathbb{R}^d)$ functions with fixed support, see Theorem 4.42 and Corollary 4.43. This yields a bounded sequence support in a fixed ball in $W^{1,p}(\mathbb{R}^d)$. By Theorem 4.45 there exists C such that

$$\|u_j(.+h) - u_j\|_{L^p(\mathbb{R}^d)} \le C|h|.$$

Now we apply the Theorem 3.39 of Kolmogorov and obtain convergence of a subsequence in $L^p(U)$. A norm limit is also a weak limit, and hence the limit is u. $\square$

[27.01.2017]

[01.02.2017]

Lemma 5.35. *Let $1 \le p < \infty$, $F : \mathbb{R}^d \to \mathbb{R}$ be nonnegative and suppose that $c \ge 1$ and*

$$F(x) \le c(1+|x|)^p \qquad \text{for all } x \in \mathbb{R}^d.$$

Suppose that $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$ and

$$F(x+h) - F(x) \ge \langle v, h\rangle \tag{5.5}$$

for all $h \in \mathbb{R}^d$. Then

$$|v| \le 2^p c(1+|x|)^{p-1}$$

Proof. There is nothing to show if $|v| \le 1$. Suppose that $|v| \ge 1$ and let $h = \frac{|x|+1}{|v|} v$. Then

$$|v|(|x|+1) \le F(x + \frac{|x|+1}{|v|} v) \le c\Big(1 + \Big|x + \frac{|x|+1}{|v|} v\Big|\Big)^p \le 2^p c(1+|x|)^p$$

and we obtain the desired conclusion. $\square$

Let $1 < p < \infty$, $U \subset \mathbb{R}^d$ open and bounded and let

$$F : U \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$$

be measurable. We assume that for all $x \in U$

$$(u, P) \to F(x, u, P)$$

is continuous and for every $x \in U$ and $u \in \mathbb{R}$ the map

$$P \to F(x, u, P)$$

is convex. We assume that there exists $C > 0$ so that

$$0 \le F(x, u, P) \le C(1 + |P|^p) \tag{5.6}$$

Theorem 5.36 (Weak lower semicontinuity)**.** *If $u \in W^{1,p}(U)$ then $F(x,u(x),\nabla u(x))$ is measurable and integrable. Moreover, if $u_n \rightharpoonup u$ in $W^{1,p}(U)$ then*

$$E(u) \le \liminf_n E(u_n) \qquad \textit{with } E(u) = \int_U F(x,u(x),\nabla u(x))\, dm^d(x).$$

Proof. We prove the result under the additional assumption that

$$P \to F(x,u,P)$$

is differentiable for all x and that

$$(u,P) \to \frac{\partial F}{\partial P_j}(x,u,P)$$

is continuous for all x and j.

Measurability and integrability follows as in Subsection 4.8. Let u_n be as in the assumptions. By Lemma 5.34

$$u_n \to u \qquad \text{in } L^p(U)$$

and by the proof of Theorem 3.17 a subsequence converges almost everywhere. Without relabelling we assume that u_n converges almost everywhere.

We write

$$E(u_n)-E(u) = \int_U F(x,u_n,\nabla u_n) - F(x,u_n,\nabla u) + F(x,u_n,\nabla u) - F(x,u,\nabla u) dm^d$$

Then

$$F(x,u_n(x),\nabla u(x)) \to F(x,u(x),\nabla u(x))$$

almost everywhere. Moreover

$$C(1+|\nabla u(x)|^p)$$

is majorant and by the convergence theorem of Lebesgue

$$\int F(x,u_n,\nabla u) - F(x,u,\nabla u) dm^d \to 0.$$

For every x the map

$$P \to F(x,u_n(x),P)$$

is convex. Hence

$$\begin{aligned} &F(x,u_n(x),\nabla u_n(x)) - F(x,u_n(x),\nabla u(x)) \\ &\ge \frac{\partial F}{\partial P}(x,u_n(x),\nabla u(x)) \cdot (\nabla u_n(x) - \nabla u(x)). \end{aligned}$$

By Lemma

$$\left|\frac{\partial F}{\partial P_j}\right| \le c(1+|\nabla u(x)|^{p-1}).$$

We observe that

$$\frac{\partial F}{\partial P_j}(x,u_n(x),\nabla u(x)) \to \frac{\partial F}{\partial P_j}(x,u(x),\nabla u(x))$$

for almost all x. Moreover

$$\left|\frac{\partial F}{\partial P_j}(x,u_n,\nabla u(x)) - \frac{\partial F}{\partial P_j}(x,u,\nabla u(x))\right| \le c(1+|\nabla u(x)|^{p-1})$$

and hence by Lebesgue

$$\left\|\frac{\partial F}{\partial P_j}(x,u_n,\nabla u(x)) - \frac{\partial F}{\partial P_j}(x,u,\nabla u(x))\right\|_{L^{\frac{p}{p-1}}} \to 0.$$

Since

$$\nabla u_n - \nabla u \rightharpoonup 0 \qquad \text{in } L^p$$

we obtain

$$\begin{aligned}&\lim_{n\to\infty}\int \frac{\partial F}{\partial P_j}(x,u_n,\nabla u(x))\partial_j(u_n-u)dm^d(x)\\ &= \lim_{n\to\infty}\int \frac{\partial F}{\partial P_j}(x,u,\nabla u(x))\partial_j(u_n-u)dm^d(x) = 0.\end{aligned}$$

□

Essentially without changing the proof we may relax (5.6) by

$$-C|u| \le F(x,u,P) \le C(1+|P|^p+|u|^p)+f(x) \tag{5.7}$$

for some integrable function f.

Theorem 5.37. *Suppose that in addition to the assumptions of Theorem 5.36 the function F is coercive: There exists $C>0$ sich that*

$$F(x,u,P) \ge C|P|^p.$$

Let $g \in W^{1,p}(U)$. Then there exists u which satisfies

$$u-g \in W_0^{1,p}(U)$$

and

$$E(u) = \inf\{E(v) : v-g \in W_0^{1,p}(U)\}.$$

Proof. Let

$$\mathcal{A} = \{u \in W^{1,p}(U) : u - g \in W_0^{1,p}(U)\}.$$

Let u_j be a minimizing sequence. Then

$$E(u_j) \geq C \int |\nabla u_j|^p dm^d,$$

and hence by Poincaré's inequality

$$\begin{aligned} \|u_j\|_{W^{1,p}} \leq & \|u_j\|_{L^p} + \|\nabla u_j\|_{L^p} \\ \leq & \|g\|_{L^p} + \|u_j - g\|_{L^p} + \|\nabla u_j\|_{L^p} \\ \leq & \|g\|_{L^p} + d\|\nabla(u_j - g)\|_{L^p} + \|\nabla u_j\|_{L^p} \\ \leq & c\left(\|g\|_{W^{1,p}} + E(u_j)^{\frac{1}{p}}\right) \end{aligned}$$

and hence

$$\sup_j \|u_j\|_{W^{1,p}(U)} < \infty.$$

By Theorem 5.36 $u_j - g$ has a weakly convergent subsequence, which converges in $L^p(U)$, and thus the same is true for u_j. Thus

$$E(u) \leq \liminf E(u_j) = \inf\{E(v) : v \in \mathcal{A}\}$$

and $u \in \mathcal{A}$ since subspaces are convex. □

Theorem 5.38 (Euler-Lagrange equations)**.** *Suppose in addition that for every x, F is continuously differentiable with respect to u and P and that*

$$\left|\frac{\partial F}{\partial u}(x, u, P)\right| \leq c(1 + |P|)^p. \tag{5.8}$$

Let u be a minimizer in $\mathcal{A}$. Then u is a weak solution to

$$-\sum_{j=1}^d \partial_{x_j} \frac{\partial F}{\partial P_j}(x, u, \nabla u) + \frac{\partial F}{\partial u}(x, u, \nabla u) = 0.$$

Proof. We argue as for the Dirichlet integral, choose $v \in \mathcal{D}(U)$ and define $\eta(t) = E(u+tv)$. Then $\eta(t) \geq \eta(0) = E(u)$. We claim that it is differentiable with respect to t:

$$\frac{1}{t}(F(x, u+tv, \nabla u+t\nabla v) - F(x, u, \nabla u)) \to \sum_{j=1}^d \frac{\partial F}{\partial P_j}(x, u, \nabla u)\partial_j v + \frac{\partial F}{\partial u}(x, u, \nabla u)v$$

almost everywhere. Moreover

$$\left|\frac{1}{t}(F(x, u + tv, \nabla u + t\nabla v) - F(x, u, \nabla u + t\nabla v))\right| \leq c(1 + |\nabla u| + |\nabla v|)^p$$

by the fundamental theorem of calculus and assumption (5.8). Moreover

$$\left|\frac{1}{t}(F(x,u,\nabla u + t\nabla v) - F(x,u,\nabla u))\right| \le c(1+|\nabla u|+|\nabla v|)^{p-1}$$

by the fundamental theorem of calculus and Lemma 5.35. Then by the convergence theorem of Lebesgue

$$0 = \eta'(0) = \int \frac{\partial F}{\partial P_j}(x,u,\nabla u)\partial_j v + \frac{\partial F}{\partial u}(x,u,\nabla u) v dm^d$$

□

Lemma 5.39. *The minimizer is unique if in addition*

$$(u,P) \to F(x,u,P)$$

is striclty convex for every x.

We call F strictly convex if for all $0 < \lambda < 1$ and x, y

$$\lambda F(x) + (1-\lambda)F(y) > F(\lambda x + (1-\lambda)y).$$

Proof. We claim that

$$\mathcal{A} \ni u \to E(u)$$

is strictly convex. It suffices to verify that

$$E(\frac{u+v}{2}) < \frac{1}{2}(E(u)+E(v))$$

unless $u = v$ which is equivalent to

$$\int_{\mathbb{R}^d} F(x,u,\nabla u) + F(x,v,\nabla v) - 2F(x, \frac{u+v}{2}, \frac{\nabla u + \nabla v}{2}) dm^d > 0.$$

The integrant is nonnegative, and the integral is positive unless it vanishes almost everywhere. But this is equivalent to $u = v$. □

[01.02.2017]

[03.02.2016]

6 Linear Operators

6.1 Dual and adjoint operators

Definition 6.1. *Let X and Y be Banach spaces, $T \in L(X,Y)$. We define the dual operator $T^{'} \in L(Y^*, X^*)$ by*

$$(T^{'} y^*)(x) = y^*(Tx) \qquad \textit{for } y^* \in Y^*, x \in X.$$

This formula defines a unique map $X \to \mathbb{K}$. It is clearly linear and bounded, hence it is in X^*.

Examples and properties

1. $1 \le p < \infty$, $X = Y = L^p(\mu)$, $g \in L^\infty$, $Tf = gf$, $T' : L^{\frac{p}{p-1}} \to L^{\frac{p}{p-1}}$, $T'h = gh$.

2. $X = Y = l^p(\mathbb{Z})$, $T : X \to X$ the shift operator, $Te_j = e_{j+1}$. Then $T'e_j = e_{j-1}$.

3. $X = Y = l^p(\mathbb{N})$, $T : X \to X$ the shift operator, $Te_j = e_{j+1}$. Then $T'e_j = e_{j-1}$ if $j \ge 2$ and $T'e_1 = 0$.

4. $J : X \to X^{**}$, $J' : X^{***} \to X^*$ is the restriction, if we identify X with a closed subspace of X^{**}.

5. Consider $\mathbb{K} = \mathbb{R}$,
$$-\Delta u = f \qquad u|_{\partial U} = 0$$
$$T : L^2 \ni f \to u \in L^2$$
Then $T' = T$.

6. $(\lambda T + \mu S)' = \lambda T' + \mu S'$, i.e. the map $T \to T'$ in complex linear.

7. $\|T'\|_{Y^* \to X^*} = \|T\|_{X \to Y}$

Definition 6.2. *Let H_1 and H_2 be Hilbert spaces, $T \in L(H_1, H_2)$. The adjoint operator is defined by*
$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

Definition 6.3. *Let $T \in L(X, Y)$. We define its range* $\operatorname{ran} T$ *as*
$$\operatorname{ran} T = \{y \in Y : \textit{ there exists } x \textit{ with } T(x) = y\} \subset Y$$
and
$$\ker T = \{x \in X : Tx = 0\}.$$
If $X_0 \subset X$ is a subspace then
$$X_0^\perp := \{x^* \in X^* : x^*(x) = 0 \quad \textit{for } x \in X_0\}$$
and if $X_0^* \subset X^*$ is a subspace then
$$(X_0^*)_\perp := \{x \in X : x^*(x) = 0 \quad \textit{for all } x^* \in X_0^*\}.$$

Lemma 6.4. *Let X and Y be Banach spaces and $T \in L(X, Y)$. Then*
$$\overline{\operatorname{ran} T} = (\ker T')_\perp$$

Proof. Let $Tx_j \to y \in \operatorname{ran} T$ and $y^* \in Y^*$ with $T'y^* = 0$. Then

$$y^*(y) = \lim_{j\to\infty} y^*(Tx_j) = \lim_{j\to\infty} T'y^*(x_j) = 0.$$

Thus

$$\overline{\operatorname{ran} T} \subset (\ker T')_\perp.$$

Now suppose that

$$y \in (\ker T')_\perp$$

By Hahn Banach there exists y^* such that $y^*|_{\operatorname{ran} T} = 0$ and

$$y^*(y) = \operatorname{dist}(y, \operatorname{ran} T).$$

But $y^* \in \ker T'$ since $y^*|_{\operatorname{ran} T} = 0$, hence $y^*(y) = 0 = \operatorname{dist}(y, \operatorname{ran} T)$. $\square$

Corollary 6.5. *Suppose that $T \in L(X,Y)$ has closed range. Then*

$$Tx = f$$

is sovable if and only if

$$T'y^* = 0 \Longrightarrow y^*(f) = 0.$$

Theorem 6.6. *Let X and Y be Banach spaces and $T \in L(X,Y)$. The following statements are equivalent:*

1. $\operatorname{ran} T$ *is closed,*
2. $\operatorname{ran} T'$ *is closed,*
3. $\operatorname{ran} T = (\ker T')_\perp$,
4. $\operatorname{ran} T' = (\ker T)^\perp$.

Proof. **Step 1: '1 $\Longleftrightarrow$ 2'**
Let $\tilde{y}^* \in (\operatorname{ran} T)^*$. By Hahn-Banach we may extend it to $y^* \in Y^*$. Since

$$T'y^*(x) = y^*(Tx) = \tilde{y}^*(Tx) := T_1'\tilde{y}^*(x),$$

the range of $T' \in L(Y^*, X^*)$ and $T_1' \in L(\overline{\operatorname{ran} T}^*, X^*)$ are the same. Without loss of generality we may assume that $\overline{\operatorname{ran} T} = Y$.
Step 1a: Now suppose that $\operatorname{ran} T$ is closed. We want to prove that $\operatorname{ran} T'$ is closed. By the consideration above it suffices to consider the case that T is surjective. By the open mapping theorem Theorem 4.3 there exists c so that for every $y \in Y$ there exists $x \in X$ with

$$\|x\|_X \le c\|y\|_Y \quad \text{and } Tx = y.$$

Thus

$$|y^*(y)| = |y^*(Tx)| = |T'y^*(x)| \le \|x\|_X \|T'y^*\|_{X^*} \le c\|y\|_Y \|T'y^*\|_{X^*}$$

and

$$\|y^*\|_{Y^*} \le c\|T'y^*\|_{X^*}$$

and hence $\operatorname{ran} T'$ is closed.

Step 1b: Now suppose that $\operatorname{ran} T'$ is closed. We want to prove that $\operatorname{ran} T$ is closed. Again it suffices to consider the case that $\operatorname{ran} T$ is dense in Y.

Suppose there is sequence $y_n \to 0$ such that $y_n \notin \overline{T(B_1(0))}$. By the separation theorems there exists $y_n^* \in Y^*$ with $\|y_n^*\|_{Y^*} = 1$ such that

$$\operatorname{Re} y_n^*(y_n) > \sup_{\|x\|_X \le 1} \operatorname{Re} y_n^*(Tx) = \sup_{\|x\|_X \le 1} \operatorname{Re} T'y_n^*(x).$$

Thus

$$\|T'y_n^*\|_{X^*} \le \|y_n\|_Y \|y_n^*\|_{Y^*} = \|y_n\|_Y \to 0.$$

However, by the Open Mapping Theorem 4.3, there exists $c > 0$ such that if $x^* \in \operatorname{ran} T'$ there exists $y^* \in Y^*$ so that

$$\|y^*\|_{Y^*} \le c\|x^*\|_{X^*}, \quad T'y^* = x^*.$$

However T' is injective since $\overline{\operatorname{ran} T} = Y$ by Lemma 6.4. Hence

$$1 = \|y_n^*\|_{Y^*} \le c\|T'y_n^*\|_{X^*} \to 0.$$

This is a contradiction. Thus there exists $r > 0$ so that

$$B_r^Y(0) \subset \overline{T(B_1(0))}.$$

Let $y \in B_r^Y(0)$. There exists x_0 with $\|x_0\|_X < 1$ and $\|y - Tx_0\|_Y \le r/2$. Then there exists x_1 with $\|x_1\|_X < \frac{1}{2}$ and $\|y - T(x_0 + x_1)\|_Y < 2^{-2}r$. Recursively we obtain a sequence x_j with $\|x_j\|_X < 2^{-j}$ and

$$\|y - T\sum_{n=0}^{N-1} x_n\|_Y < 2^{-N}r.$$

Then

$$y = T\sum_{n=0}^{\infty} x_n$$

and the range intersected with $\overline{B_r^Y(0)}$ is closed. Hence the range is closed.

Step 2: '1 $\iff$ 3':

This is a consequence of Lemma 6.4.

Step 3: '2 $\iff$ 4:

'2 $\Longleftarrow$ 4' is trivial since $(\ker T)^\perp$ is closed.

So assume that $\operatorname{ran} T'$ is closed and hence $\operatorname{ran} T$ is closed. It is obvious that $\operatorname{ran} T' \subset (\ker T)^\perp$. Let $x^*|_{\ker T} = 0$. If $y = Tx$ we define

$$y^*(y) = x^*(x)$$

This is well defined since x^* vanishes on the null space of T. Since we may assume that T is surjective we obtain from the open mapping theorem that for $y \in Y$ there exists $x \in X$ with $Tx = y$ and $\|x\|_X \le C\|y\|_Y$. Thus

$$|y^*(y)| \le C\|x^*\|_{X^*}\|y\|_Y$$

and there is a unique continuous extension and we may consider $y^* \in Y^*$. But then $x^* = T'y^* \in \operatorname{ran} T'$. □

[03.02.2016]

[08.02.2017]

Theorem 6.7. *If $T \in L(X,Y)$ is invertible iff $T' \in L(Y^*, X^*)$ is invertible.*

Proof. Suppose that T is invertible. Thus the range is closed. By Theorem 6.6 also $\operatorname{ran} T'$ is closed. Again by Lemma 6.6 T' is injective since $Y = \operatorname{ran} T = (\ker T')_\perp$. Moreover $\operatorname{ran} T' = (\ker T)^\perp = X^*$. The same argument gives the reverse implication.

□

6.2 Compact operators

Definition 6.8. *Let X and Y be Banach spaces, $T \in L(X,Y)$. We call T compact if for every bounded sequence (x_j), (Tx_j) contains a convergent subsequence.*

Lemma 6.9. *The following are equivalent*

1. *T is compact.*
2. *The image of the closed unit ball is relatively compact (the closure is compact).*
3. *The image of a bounded set is relatively compact.*
4. *The image of a bounded set is precompact.*

Proof. Suppose that T is compact. Let $K = T(\overline{B_1(0)})$. Let $x_j \in B_1^X(0)$ and $y_j = Tx_j$. Then (x_j) is a bounded sequence and since T is compact (Tx_j) contains a convergent subsequence. Thus the image of the ball of radius 1 is relatively compact.

The second statement obviously implies the third. The closure of a relatively compact set is compact, and hence precompact. Now assume that $TB_1^X(0)$ is precompact. Then the closure is compact, and if (x_j) is a sequence with $\|x_j\|_X < 1$ then (Tx_j) is a sequence in a precompact set, and hence there is a convergent subsequence and T is compact. $\square$

Lemma 6.10. *If T is compact, S and U are continuous, then STU is compact. If $(T_j) \in L(X,Y)$ are compact and*

$$T_j \to T \qquad \text{in } L(X,Y)$$

then T is compact. If T has finite rank it is compact. If an invertible map is compact then X and Y are finite dimensional.

Proof. Let $S \in L(Y,Z), T \in L(X,Y)$ and $U \in (V,X)$ as above and (v_j) a bounded sequence in V. Then (Uv_j) is a bounded sequence in X since U is bounded, (TUv_{j_l}) is a converging subsequence in Y since T is compact and $(STUx_{j_l})$ is a convergent sequence in Z since S is continuous.

Let $T_j \to T$ a convergent sequence of compact operators and let (x_n) be a bounded sequence. For simplicity we assume that it is bounded by 1.

We claim that for every $\varepsilon > 0$ there exists a subsequence x_{n_j} so that

$$\|Tx_{n_j} - Tx_{n_l}\|_Y < \varepsilon.$$

Suppose the claim is true. We apply it iteratively with $\varepsilon = 2^{-j}$. We choose $\tilde{x}_n$ then from the nth iteration. Then $T\tilde{x}_n$ is a Cauchy sequence.

Let $\varepsilon > 0$. There exists n_0 so that $\|T - T_n\|_{X \to Y} < \varepsilon/3$ for $n \geq n_0$. Since T_{n_0} is compact there exists a subsequence so that $T_{n_0}x_{n_j}$ is a Cauchy sequence and l_0 so that

$$\|T_{n_0}(x_{j_l} - x_{j_m})\|_Y < \varepsilon/3$$

for $l, m \geq l_0$. Thus

$$\|T(x_{j_l} - x_{j_m})\|_Y < \varepsilon$$

for $l, m \geq l_0$.

Now let T be an operator of finite rank. Let (x_j) be a bounded sequence. Then (Tx_j) is a bounded sequence in a finite dimensional space and there exists a convergent subsequence.

An invertible linear operator is compact if and only if every bounded sequence has a convergent subsequence, which by Heine-Borel holds iff the space has finite dimension.

$\square$

Lemma 6.11. *Let $T \in L(X,X)$ be compact. Then* $\operatorname{ran}(1_X - T)$ *is closed.*

Proof. Suppose that $(x_j - Tx_j)$ is a convergent sequence where $\|x_j\|_X = 1$. Then

$$x_j = Tx_j + (x_j - T(x_j))$$

The second term converges by assumption and the first has a convergent subsequence. Without changing the notation we assume that $x_j \to x$. Then

$$x_j - Tx_j \to x - Tx \in \operatorname{ran}(1-T).$$

Adding an element of the null space $\ker(1-T)$ we may assume that

$$\|x_n\| \le 2\operatorname{dist}(x_n, \ker(1-T)).$$

If $\|x_n\|$ is unbounded, there is a subsequence so that

$$\operatorname{dist}(x_{n_j}, \ker(1-T)) \ge n_j.$$

Let $\tilde{x}_j = \frac{1}{\|x_{n_j}\|} x_{n_j}$. Then

$$\tilde{x}_j - T\tilde{x}_j = \frac{x_{n_j} - Tx_{n_j}}{\|x_{n_j}\|} \to 0$$

and, since $\|\tilde{x}_j\| = 1$ there is another subsequence $T\tilde{x}_{j_l}$ converges to $\tilde{x}^\infty \in X$

$$\tilde{x}_{j_l} = \tilde{x}_{j_l} - T\tilde{x}_{j_l} + T\tilde{x}_{j_l} \to \tilde{x}^\infty.$$

Then $\|\tilde{x}^\infty\|_X = 1$, $\operatorname{dist}(\tilde{x}^\infty, \ker(1-T)) \ge \frac{1}{2}$ which is a contradiction to $\tilde{x}^\infty = T\tilde{x}^\infty$. □

Theorem 6.12 (Schauder)**.** *Let X, Y be Banach spaces, $T \in L(X,Y)$.*

$$T \quad \textit{compact} \qquad \Longleftrightarrow \qquad T' \quad \textit{compact}$$

Proof. Let T be compact, $y_j^* \in Y^*$ a bounded sequence and

$$K = \overline{TB_1(0)} \subset Y.$$

K is compact since T is compact and the functionals y_j^* define uniformly continuous bounded functions on K. By the theorem of Arzela Ascoli 3.34 there exists a convergent subsequence $y_{j_l}^*$. But then

$$T'y_{j_l} \in X^*$$

is a Cauchy sequence. If T' is compact then also T'' and also T since $J_Y T = T'' J_X$. □

Lemma 6.13. *Suppose that $T \in L(X)$ is compact and* $\ker(1-T) = \{0\}$. *Then there exists $\varepsilon > 0$ so that*

$$\varepsilon \|x\| \le \|x - Tx\|.$$

Proof. If not there exists (x_n) with $\|x_n\| = 1$ and $\|x_n - Tx_n\| < \frac{1}{n}\|x_n\|$. There is a subsequence and $x_\infty \in X$ with $Tx_{n_j} \to x_\infty$. Then

$$x_{n_j} = (x_{n_j} - Tx_{n_j}) + Tx_{n_j} \to x_\infty$$

and $\|x_\infty\|_X = 1$ and $x_\infty - Tx_\infty = 0$. This is a contradiction with the assumption of the lemma. □

Lemma 6.14. *Let X be a Banach space, $T \in L(X)$ such that $\ker T = \{0\}$ and T is not surjective. Then $\operatorname{ran} T^{n+1} \subset \operatorname{ran} T^n$ but $\operatorname{ran} T^{n+1} \neq \operatorname{ran} T^n$.*

Proof. We have always

$$\operatorname{ran} T^{n+1} = T^n(TX) \subset \operatorname{ran} T^n X.$$

Since $\operatorname{ran} T \neq X$ there exists $x \notin \operatorname{ran} T$. If $T^n x = T^{n+1} y$ for some $y \in X$ then

$$T^n(x - Ty) = 0$$

and since T is injective $x = Ty$, a contradiction. □

Lemma 6.15. *Let $T \in L(X)$ be compact. If $\ker(1 - T) = \{0\}$ then $\operatorname{ran}(1 - T) = X$ and $1 - T$ is invertible.*

Proof. Let $S = 1 - T$. Then

$$S^n = (1 - T)^n = \sum_{j=0}^{n} \binom{n}{j} (-T)^j = 1 + \sum_{j=1}^{n} \binom{n}{j} (-T)^j = 1 - \tilde{T}$$

with $\tilde{T}$ compact. Then $\ker S^n = \{0\}$ and the range of S^n is closed by Lemma 6.11. If S is not surjective then by Lemma 6.14 there exists a sequence y_n so that $\|y_n\| = 1$, $y_n \in \operatorname{ran} S^n$ and

$$\operatorname{dist}(y_n, \operatorname{ran} S^{n+1}) \geq \frac{1}{2}.$$

We claim that

$$\|Ty_n - Ty_m\| \geq \frac{1}{2}$$

if $n < m$. This is a contradiction to compactness and it remains to prove the claim. It follows from

$$Ty_n - Ty_m = y_n - y_m - S(y_n - y_m)$$

and

$$y_m + S(y_n - y_m) \in \operatorname{ran} S^{n+1}.$$

Thus S is injective and surjective, and invertibility follows now from the theorem of the inverse, Corollary 4.4. □

Theorem 6.16 (Riesz-Schauder)**.** *Let $T \in L(X, X)$ be compact. $\lambda \neq 0$ is an eigenvalue of T iff it is an eigenvalue of T'.*

Proof. By Lemma 6.15 $T - 1$ is invertible iff 1 is not an eigenvalue. By Theorem 6.12 T' is compact iff T is compact. By Theorem 6.7 $T - 1$ is invertible iff $T' - 1_{X^*}$ is invertible. Again by Lemma 6.15 $T' - 1$ is invertible iff 1 is not an eigenvalue of T'. This completes the proof since $T - \lambda 1$, $\lambda \neq 0$ is invertible iff $\lambda^{-1}T - 1$ is invertible. □

Theorem 6.17 (Fredholm alternative)**.** *Let $T : X \to X$ be compact. Then either for $f \in X$ the problem*

$$x - Tx = f$$

has a unique solution or

$$x - Tx = 0$$

has a nontrivial solution.

In general

$$x - Tx = f$$

is solvable iff $x^*(f) = 0$ for all $x^* \in \ker T'$.

Proof. If $x - Tx = 0$ has a nontrivial solution and if $x - Tx = f$ has a solution then it has infinitely many solutions. If $x - Tx = 0$ has no nontrivial solution then $1 - T$ is invertible by Lemma 6.15. The last statement follows from Lemma 6.6. □

[08.02.2017]

[10.02.2017]

6.2.1 Examples of compact operators

Let $1 \leq p \leq \infty$ and $X = l^p(\mathbb{N}; \mathbb{C})$, $(a_n) \in c_0$. We define $T \in L(X)$ by

$$T((x_n)_{n\in\mathbb{N}}) = (a_n x_n)_{n\in\mathbb{N}}.$$

We claim that T is compact. Let $a_n^N = a_n$ if $n \leq N$ and $a_n^N = 0$ otherwise and $T_N(x_n) = (a_n^N x_n)$. Then $\operatorname{rk} T_N \leq N$, hence T_N is compact and $T_N \to T$ in $L(X)$. Thus T is compact by Lemma 6.10.

Lemma 6.18 (Compact embeddings)**.** *Let U be bounded and $1 \leq p < \infty$. Then the embedding $W_0^{1,p}(U) \to L^p(U)$ is compact. The same is true for the embedding $W^{1,p}(U) \to L^p(U)$ under the Whitney extension assumption.*

Proof. In either case there is an extension operator. We focus on $W_0^{1,p}(U)$:

$$E : W_0^{1,p}(U) \to W^{1,p}(\mathbb{R}^d)$$

such that every function in the range is supported in a fixed ball. By the difference characterisation of Sobolev functions

$$\sup_{\|f\|_{W^{1,p}(U)}\le 1} \sup_{h\neq 0} \frac{1}{|h|} \|Ef(.+h) - Ef\|_{L^p(\mathbb{R}^d)} \le c\|f\|_{W_0^{1,p}(U)}$$

and

$$\sup_{\|f\|_{W^{1,p}(U)}\le 1} \|Ef\|_{L^p(\mathbb{R}^d)} \le c\|f\|_{W_0^{1,p}(U)}.$$

By Theorem 3.39 and Corollary 4.46 the unit ball in $W^{1,p}(U)$ is compact in $L^p(U)$. □

Lemma 6.19. *Let U be open and bounded. If $f \in L^2(U)$ there exists a unique $u \in W_0^{1,2}(U)$ which satisfies*

$$-\Delta u = f$$

as distribution. The map $f \to u$ defines a compact operator

$$T : L^2(U) \to L^2(U).$$

If $z \notin (0,\infty)$ the equation

$$-\Delta u = zu + f \tag{6.1}$$

has a unique solution in $W_0^{1,2}(U)$ for all $f \in L^2(U)$.

Proof. By the Lemma of Lax-Milgram, the operator $\tilde T : L^2(U) \ni f \to u \in W_0^{1,2}(U)$ is continuous. Hence the composition of $\tilde T$ with the compact embedding $W_0^{1,2}(U) \to L^2(U)$ is compact. That is, $T : L^2(U) \ni f \to u \in L^2(U)$ is compact.

We rewrite (6.1) as

$$u - zTu = \tilde f := Tf. \tag{6.2}$$

Since T is compact, (6.2) is uniquely solvable for every $f \in L^2(U)$ iff $u - zTu = 0$ has only the trivial solution. This equation $u = T(zu)$ can be reformulated as weak solution to

$$-zu - \Delta u = 0$$

which is equivalent to

$$0 = \int \sum_{j=1}^d \partial_{x_j} u \overline{\partial_{x_j} v} - zu\bar v \, dm^d$$

which with $v = u$ gives

$$0 = \int |Du|^2 dm^d - z \int |u|^2 dm^d.$$

This equation has only the trivial solution $u = 0$ unless $z \in (0, \infty)$. Thus (6.2) is uniquely solvable and the solution is the unique weak solution to (6.1).

$\square$

Finally let $X = L^2([0,1])$ and

$$(Tf)(x) = \int_0^x f(y) dy, \quad x \in [0,1].$$

Then

$$T : L^2((0,1)) \to W^{1,2}((0,1))$$

and it is compact as an operator to $L^2((0,1))$ since the embedding is compact. We claim that there is no eigenvalue. If

$$zf(x) = \int_0^x f(y) dy$$

then, if $z \neq 0$

$$f' = z^{-1} f, \qquad f(0) = 0.$$

By Gronwall's lemma

$$|f(x)| \le e^{|z|^{-1} x} |f(0)| = 0.$$

Hence $f = 0$ and $z \neq 0$ is not an eigenvalue. On the other hand if $z = 0$ then

$$\int_0^x f(y) dy = 0.$$

This implies that any antiderivative of f is constant, and hence $f = 0$. We obtain

Lemma 6.20. *The operator T is compact. It has no eigenvalues.*

6.2.2 Eigenvalues and spectrum

Lemma 6.21. *The set of invertible operators in $L(X,Y)$ is open.*

Proof. Suppose that $T \in L(X,Y)$ is invertible. Take $S \in L(X,Y)$ with $\|S\|_{L(X,Y)} \|T^{-1}\|_{L(Y,X)} < 1$. We define

$$A = \sum_{n=0}^{\infty} (T^{-1} S)^n T^{-1}$$

Since

$$\|(T^{-1}S)^nT^{-1}\|_{L(Y,X)} \le (\|T^{-1}\|_{L(Y,X)}\|S\|_{L(X,Y)})^n\|T^{-1}\|_{L(Y,X)}$$

the partial sums converge. A straightforward calculation shows that

$$(T-S)A = 1_Y, \qquad A(T-S) = 1_X.$$

Thus the operator $T-S$ is invertible and hence the set of invertible operators is open. □

Theorem 6.22 (Spectrum of compact operators, Riesz-Schauder)**.** *Let $T \in L(X)$ be compact and $0 \neq \lambda$. If λ is not an eigenvalue then $T-\lambda$ is invertible. Suppose that λ is an eigenvalue, then there exists $k_0 \in \mathbb{N}$ such that*

1. $\ker(T-\lambda)^k = \ker(T-\lambda)^{k_0} =: N$ *if* $k \ge k_0$.
2. $\operatorname{ran}(T-\lambda)^k = \operatorname{ran}(T-\lambda)^{k_0} =: R$ *if* $k \ge k_0$.
3. $T : N \to N$ *and* $T : R \to R$. *The second map is invertible.*
4. $\dim N = \dim\ker(T'-\lambda)^{k_0} < \infty$.
5. *Every* $x \in X$ *can uniquely be written as* $x = y + z$ *with* $y \in N$ *and* $z \in R$.
6. 0 *is the only possible accumulation point of the eigenvalues.* $T-\lambda$ *is invertible if* λ *is neither an eigenvalue nor zero.*

Proof. Let λ be an eigenvalue. We claim that $\dim\ker(T-\lambda) < \infty$. If the eigenspace of the eigenvalue λ is infinite dimensional, then the closed ball in the eigenspace with radius $|\lambda|$ - the image of the closed ball with radius 1 under T - is not compact.

Let $R_k = \operatorname{ran}(T-\lambda)^k$. Then there exists k_0 so that $R_k = R_{k_0}$ if $k > k_0$. We argue as in Lemma 6.15.

Thus $\ker(T'-\lambda)^k = \ker(T'-\lambda)^{k_0}$. Reversing the role of T and T' we see that there exists k_0 so that also $\ker(T-\lambda)^k = \ker(T-\lambda)^{k_0}$ for $k \ge k_0$.

Let $x \in N$, then $(T-\lambda)^{k_0}Tx = T(T-\lambda)^{k_0}x = 0$ and hence $Tx \in N$. If $x \in R$ then $(T-\lambda)x \in (T-\lambda)R = R$ and hence $Tx \in R$.

If $x \in R = \operatorname{ran}(T-\lambda)^{k_0}$ then there exists y such that $(T-\lambda)^{k_0}y = x$. Suppose that also $x \in N = \ker(T-\lambda)^{k_0}$. Then

$$(T-\lambda)^{2k_0}y = 0 = (T-\lambda)^{k_0}y = x.$$

Thus $R \cap N = \{0\}$ and $T : R \to R$ is injective and surjective, and hence invertible.

We apply the same reasoning to $(T'-\lambda)^{k_0}$ and obtain N' and R' in X^*. Then $N' \to N'|_N \in N^*$ is injective (otherwise there would be a nontrivial element in $N' \cap R'$). Similarly $N \to (N')^*$ is injective (otherwise $N \cap R$ would be nontrivial). Since $\dim N$ and $\dim N' < \infty$ we must have $\dim N = \dim N'$ and $N' \to N'|_N$ is bijective.

Now let $x \in X$. Then there exists $n \in N$ so that $x^*(x - n) = 0$ for all $x^* \in N'$. But then $x - n \in R$.

We denote by $T_R : R \to R$ and $T_N : N \to N$. Then $T_R - \mu 1_R$ is invertible for $|\mu - \lambda|$ small. Moreover $T_N - \mu 1_N$ is invertible for $\mu \neq \lambda$ since $T_N - \lambda$ is nilpotent. Thus $T - \mu$ is invertible for $|\mu - \lambda|$ small $\mu \neq \lambda$. Also $T - \mu$ is invertible unless μ is an eigenvalue.

□

This we can apply to problem (6.1) and conclude that there is a monotone sequence of eigenvalues, with ∞ as only possible limit, and (6.1) is uniquely solvable unless z is an eigenvalue.