Prof. Dr. Herbert Koch
Dr. Xian Liao
Winter Term 2016/2017

universitätbonn

Functional Analysis and Partial Differential Equations

Sheet Nr.8 Due: 16.12.2016

Exercise 1

Let $X = C_b([0,1])$.

a) Let X_n be the set of continuous piecewise affine functions f whose derivative satisfies $|f'(x)| \geq n$ almost everywhere (i.e. $f \in X_n$ if there exists $0 = t_0 < t_1 < \cdots < t_N = 1$ so that f is affine on $[t_j, t_{j+1}]$ with $|f'(t)| \geq n$ for $t \in (t_j, t_{j+1})$). Prove that $X_n \subset C_b([0,1])$ is dense.

b) Let
$$Y_n = \{f \in C_b([0,1]) : \text{ there exists } x_0 \text{ so that } |f(x) - f(x_0)| \leq n|x - x_0|\}.$$
Prove that Y_n is closed.

c) Prove that the interior of Y_n is empty.

d) Let $A \subset C_b([0,1])$ be the set of functions which are nowhere differentiable. Then A is dense.

Exercise 2

Let $\lambda > 0$ and
$$g(x) = \frac{1}{2\lambda} e^{-\lambda|x|}.$$
Prove that
$$-\partial^2_{xx} g + \lambda^2 g = \delta_0$$
and deduce a formula for a soution to
$$-u_{xx} + \lambda^2 u = f$$
where $f \in C_b(\mathbb{R})$. Prove that $T : f \to u$ defines for $1 \leq p \leq \infty$ (by an abuse of notation) $T \in L(L^p(\mathbb{R}), L^p(\mathbb{R}))$ and estimate $\|T\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})}$ by Schur's lemma or Young's convolution estimate.

Exercise 3

Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by
$$g(t,x) = \begin{cases} \frac{1}{2} & \text{if } |x| < -t \\ 0 & \text{if } |x| \geq -t \end{cases}$$
Prove that
$$(\partial^2_{tt} - \partial^2_{xx}) g = \delta_0(t,x).$$
Deduce D'Alembert's formula.
Hint: Consider the factorisation of $(\partial^2_{tt} - \partial^2_{xx}) = (\partial_t - \partial_x)(\partial_t + \partial_x)$.

Exercise 4

Let X be a Banach space and X^* be its dual space. We say that

$$x_n^* \overset{*}{\rightharpoonup} x^*$$

weakly in X^* if

$$x_n^*(x) \to x^*(x)$$

for all $x \in X$. Prove that then the sequence (x_n^*) is bounded.