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Winter Term 2016/2017

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Functional Analysis and Partial Differential Equations

Sheet Nr.3 Due: 11.11.2016

Exercise 1

Show that $l^\infty(\mathbb{N})$ is not separable.
Hint: For real number $r \in \mathbb{R}$, consider the sequence (x_j^r) with $x_j^r = \exp(ijr)$, $j \in \mathbb{N}$, $i^2 = -1$ and prove that

$$\left\| x_j^r - x_j^{\widetilde{r}} \right\|_{l^\infty} \geq 1,$$

unless $r - \widetilde{r}$ is a multiple of 2π.

Exercise 2

Let X be a set and let $\mathcal{A} = 2^X$ be the power set of X (i.e. $\mathcal{A}$ is the set of all subsets of X).

a) Let $x_0 \in X$. Verify that the Dirac measure

$$\delta_{x_0}(A) = \begin{cases} 1 & \text{if } x_0 \in A, \\ 0 & \text{otherwise}, \end{cases} \qquad A \in \mathcal{A},$$

defines a measure on $\mathcal{A}$.

b) Verify that the counting measure

$$\mathcal{H}^0(A) = \#A \text{ which maps } A \text{ to the number of its elements},$$

is a measure on $\mathcal{A}$.

Exercise 3

Let $B_1(0) \subset \mathbb{R}^n$ be the unit ball and $s \in \mathbb{R}$. Let m^n be the Lebesgue measure on $\mathbb{R}^n$.
Evaluate

$$\int_{B_1(0)} |x|^s \, dm^n(x)$$

and

$$\int_{\mathbb{R}^n \setminus B_1(0)} |x|^s \, dm^n(x)$$

using Definition 3.3.

Exercise 4

Let $1 \leq p \leq \infty$. We want to define $T : l^p(\mathbb{Z}) \mapsto l^p(\mathbb{Z})$ by

$$T((x_j))(n) = \sum_j e^{-|j-n|} x_j.$$

Prove that $T \in L(l^p(\mathbb{Z}), l^p(\mathbb{Z}))$.

Let $p = 2$. Prove that

$$\mathrm{id} + iT, \quad i^2 = -1,$$

is invertible by using the Theorem of Lax-Milgram. Can you find an explicit formula for the inverse?