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Functional Analysis and Partial Differential Equations

Sheet Nr.2 Due: 4.11.2016

Exercise 1

Let X be a Banach space and U be a closed subvector space.
Prove that X/U is a vector space,

$$\|\widetilde{x}\| := \inf_{y \in U} \|x - y\|_X$$

defines a norm on X/U (here $\widetilde{x}$ is the equivalence class of x) and X/U is a Banach space.

Exercise 2

Let us take the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Show that H_n is a polynomial of degree n with leading term $2^n x^n$, and that for $m < n$

$$\int_{-\infty}^{+\infty} H_n(x) x^m e^{-x^2} \, dx = 0 \tag{1}$$

and

$$\int_{-\infty}^{+\infty} (H_n(x))^2 e^{-x^2} \, dx = 2^n n! \sqrt{\pi}. \tag{2}$$

Deduce that the normalized polynomials

$$\varphi_n(x) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x), \quad n = 0, 1, 2, \cdots$$

compose an orthonormal system in the Hilbert space $L^2(\mathbb{R})$, that is to show

$$\int_{-\infty}^{+\infty} \varphi_n \varphi_m \, dx = \delta_{n,m}.$$

Exercise 3

Let $L \in (l^1(\mathbb{N}))^*$ be defined by

$$L((x_j)) = \sum_{j=1}^{\infty} (1 - 2^{-j}) x_j.$$

and let

$$C := \{(x_j) \in l^1 : \mathrm{Re} L((x_j)) \geq 1\}.$$

Show that C is closed and convex. Prove that $\operatorname{dist}(0, C) = 1$ and

$$C \subset \{(x_j) \in l^1 : \|(x_j)\|_{l^1} > 1\}.$$

Exercise 4

Give examples of $T \in L(l^2(\mathbb{N}), l^2(\mathbb{N}))$ such that

a) T is injective and has closed range, but it is not surjective.

b) T is surjective but not injective.

Hint: Consider the shift operators:

$$(x_j)_{j\in\mathbb{N}} \to (x_{j+1})_{j\in\mathbb{N}}$$

and

$$(x_j)_{j\in\mathbb{N}} \to (y_j)_{j\in\mathbb{N}}$$

where $y_1 = 0$ and $y_j = x_{j-1}$ for $j \geq 2$.