V5B7: Multiple recurrence

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1 Overview

Szemerédi's theorem states that positive density subsets of integers contain arbitrarily long arithmetic progressions, that is, subsets of the form $(a, a+n, \dots, a+kb)$, $b > 0$. This can be formulated as a statement about invertible measure-preserving transformations T of a probability space (X, μ): for every positive measure subset $A \subset X$ there exists an n such that

$$\mu(A \cap T^n A \cap \cdots \cap T^{kn} A) > 0. \tag{1}$$

This phenomenon is called *multiple recurrence*: many points in the set A return to A multiple times (and the sequence of return times contains arbitrarily long arithmetic progressions). The aim of this course is to introduce the participants to the Host–Kra structure theory, which describes certain subalgebras of $L^\infty(X)$ that control the asymptotic behavior of the multicorrelation sequence (1) and its polynomial versions.

2 Topics

1. Furstenberg correspondence principle
2. Mean ergodic theorem, single recurrence, Sárközy–Furstenberg theorem
3. Conditional expectation, disintegration of measures, conditional products
4. Compact and weak mixing systems, Roth theorem
5. Uniformity seminorms
6. Compact extensions, Mackey group of a cocycle
7. Conze–Lesigne equation, nilsystems
8. Polynomials in nilpotent groups, equidistribution of polynomial orbits

3 Prerequisits

1. Measure theory (think of Riesz representation theorem, Carathéodory extension theorem)
2. Functional analysis (spectral calculus)

4 Exam

Non-binding information: oral examination (15-45 minutes, two tries, one at the end of the lecture period and one at the end of semester), no admittance requirements. See Prüfungsordnung and module handbook for official regulations:

`http://www.mathematics.uni-bonn.de/study/master/documents`

References

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[Fur77] H. Furstenberg. "Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions". In: *J. Analyse Math.* 31 (1977), pp. 204–256. MR: 0498471(58#16583).

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[Tao09] T. Tao. "Poincaré's legacies, pages from year two of a mathematical blog. Part I". Providence, RI: American Mathematical Society, 2009, pp. x+293. MR: 2523047(2010h:00003).