Analysis 1, Solutions to problem set 13

Problem 1 (Complex numbers)

Start by noticing that, for any $n \in \mathbb{N}$ and complex number $z \neq 1$,

$$1 + z + \ldots + z^n = \frac{z^{n+1} - 1}{z - 1}. \tag{1}$$

This formula follows from

$$(z-1)(1 + z + \ldots + z^n) = (z + z^2 + \ldots + z^n + z^{n+1}) + (-1 - z - \ldots - z^n) = z^{n+1} - 1$$

and can be rigorously proved via induction on n.
Now, let $0 < \theta < 2\pi$. Then $e^{i\theta} \neq 1$. The real part of the geometric sum with general term $e^{i\theta}$ is given by

$$\Re\Big(\sum_{k=0}^{n} (e^{ik\theta})^k\Big) = 1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta.$$

On the other hand, formula (1) implies that

$$\begin{aligned}\sum_{k=0}^{n} (e^{i\theta})^k = \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1} &= \frac{e^{i(n+1/2)\theta} - e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} \\ &= \frac{e^{i(n+1/2)\theta} - e^{-i\theta/2}}{2i\sin(\theta/2)} \\ &= i\frac{e^{-i\theta/2} - e^{i(n+1/2)\theta}}{2\sin(\theta/2)}.\end{aligned}$$

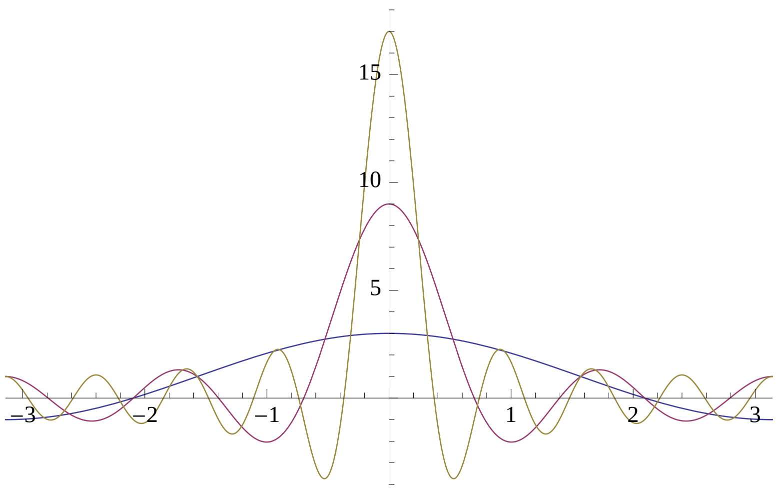
We further have that

$$\Im\Big(e^{-i\theta/2} - e^{i(n+1/2)\theta}\Big) = \sin(-\theta/2) - \sin\Big[(n+\frac{1}{2})\theta\Big],$$

whence

$$1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = \frac{\sin(\theta/2) + \sin[(n+\frac{1}{2})\theta]}{2\sin(\frac{\theta}{2})} = \frac{1}{2} + \frac{\sin[(n+\frac{1}{2})\theta]}{2\sin(\frac{\theta}{2})}.$$

Remark. The collection of functions $D_n(t) := \frac{\sin[(n+\frac{1}{2})\theta]}{\sin(\frac{\theta}{2})}$ is called the *Dirichlet kernel* and plays a prominent role in Fourier analysis. Here is a picture of the functions D_1, D_4 and D_8 (in blue, red and yellow, respectively) on the interval $[-\pi, \pi]$:



Problem 2 (Taylor's formula with the Lagrange form of the remainder)

(a) We shall proceed via induction on n. The base case $n = 0$ amounts to establishing the formula

$$f(x) = f(a) + \int_a^x f'(t)dt$$

for an arbitrary continuously differentiable function $f : I \to \mathbb{R}$, which of course follows at once form an appropriate version of the fundamental theorem of calculus. Let us now assume that the formula holds for $n-1$, i.e. with a remainder term of the form

$$R_n(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f^{(n)}(t)dt.$$

Integrating by parts, we then have that

$$\begin{aligned} R_n(x) = -\int_a^x \frac{d}{dt}\Big(\frac{(x-t)^n}{n!}\Big) f^{(n)}(t)dt &= -\frac{(x-t)^n}{n!} f^{(n)}(t)\Big|_{t=a}^{t=x} + \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt \\ &= \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt, \end{aligned}$$

and the result follows.

(b) We start by proving the following lemma, which sometimes goes in the literature under the name of *integral mean value theorem*:[1]

Lemma 1. Let $\varphi : [a, b] \to (0, \infty)$ be a nonnegative integrable function. For every continuous function $f : [a, b] \to \mathbb{R}$, there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x)\varphi(x)dx = f(\xi)\int_a^b \varphi(x)dx.$$

Proof of Lemma 1. Start by noting that the function $f\varphi$ is still integrable. Define:

$$m := \inf\{f(x) : x \in [a, b]\} \text{ and } M := \sup\{f(x) : x \in [a, b]\}.$$

It follows that $m\varphi \le f\varphi \le M\varphi$, and therefore

$$m\int_a^b \varphi(x)dx \le \int_a^b f(x)\varphi(x)dx \le M\int_a^b \varphi(x)dx.$$

As a consequence, there exists $\mu \in [m, M]$ such that

$$\int_a^b f(x)\varphi(x)dx = \mu\int_a^b \varphi(x)dx,$$

and the result then follows from the intermediate value theorem. □

The result follows at once form Lemma 1: in fact, since the corresponding assumptions are met, there must necessarily exist $\xi \in [a, x]$ (or $\xi \in [x, a]$ if $x < a$) for which

$$\begin{aligned} R_{n+1}(x) =& \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt = f^{(n+1)}(\xi)\int_a^x \frac{(x-t)^n}{n!}dt \\ =& -f^{(n+1)}(\xi)\frac{(x-t)^{n+1}}{(n+1)!}\Big|_a^x = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}. \end{aligned}$$

[1] Compare it with the differential versions from Problem 4, ÜB 11.

Problem 3 (Binomial series)

(a) Start by noting that if $\alpha \in \mathbb{N}$ the series is actually a finite sum (since in this case $\binom{\alpha}{n} = 0$ if $n > \alpha$) and the result boils down to the binomial formula.

Let us go back to the general case $\alpha \in \mathbb{R}$. Start by computing the Taylor series of the function $f(x) = (1+x)^\alpha$ around $x = 0$: since

$$f^{(k)}(x) = \alpha(\alpha-1)(\alpha-2)\cdot\ldots\cdot(\alpha-k+1)(1+x)^{\alpha-k} = k!\binom{\alpha}{k}(1+x)^{\alpha-k},$$

it follows that

$$\frac{f^{(k)}(0)}{k!} = \binom{\alpha}{k},$$

and so the Taylor series of f at 0 is given by

$$T[f,0](x) = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k. \tag{2}$$

We claim that the series (2) converges for $|x| < 1$. To see this, let us make use of the quotient criterion. For $\alpha \notin \mathbb{N}$ and $x \neq 0$, let $a_n := \binom{\alpha}{n}x^n$. We have that:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\binom{\alpha}{n+1}x^{n+1}}{\binom{\alpha}{n}x^n}\right| = |x|\left|\frac{\alpha-n}{n+1}\right|.$$

It follows that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|\lim_{n\to\infty}\left|\frac{\alpha-n}{n+1}\right| = |x| < 1,$$

and so for every θ such that $|x| < \theta < 1$, there exists n_0 for which

$$\left|\frac{a_{n+1}}{a_n}\right| \leq \theta \text{ for every } n \geq n_0.$$

It follows that the Taylor series (2) converges for $|x| < 1$.

The third and last step is to check that the Taylor series (2) converges to the value of the function f at x. By Problem 2 above, it will be enough to show that for $|x| < 1$ the remainder term $R_{n+1}(x)$ converges to 0 as $n \to \infty$. By Problem 2(a), we have that

$$R_{n+1}(x) = \frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)}(t)dt = (n+1)\binom{\alpha}{n+1}\int_0^x (x-t)^n(1+t)^{\alpha-n-1}dt.$$

Let us assume first that $0 \leq x < 1$. Set $C := \max\{1, (1+x)^\alpha\}$, and note that, for $0 \leq t \leq x$,

$$0 \leq (1+t)^{\alpha-n-1} \leq (1+t)^\alpha \leq C.$$

Consequently,

$$\begin{aligned}|R_{n+1}(x)| &= (n+1)\left|\binom{\alpha}{n+1}\right|\left|\int_0^x (x-t)^n(1+t)^{\alpha-n-1}dt\right.\\ &\leq (n+1)\left|\binom{\alpha}{n+1}\right|C\int_0^x (x-t)^n dt = C\left|\binom{\alpha}{n+1}\right|x^{n+1}.\end{aligned}$$

We have already shown that the series $\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k$ converges for $|x| < 1$, and so

$$\lim_{k\to\infty}\left|\binom{\alpha}{k}\right|x^k = 0.$$

It follows that

$$\lim_{n\to\infty} R_{n+1}(x) = 0,$$

as desired. The case $-1 < x < 0$ can be handled with minor modifications only and is therefore omitted.

(b) We start by proving an auxiliary result which characterizes the asymptotic behavior of the sequence $|\binom{\alpha}{n}|$:

Lemma 2. Let $\alpha \in \mathbb{R} \setminus \mathbb{N}$. Then there exists a constant $c = c(\alpha) > 0$ such that [2]

$$\left|\binom{\alpha}{n}\right| \sim \frac{c}{n^{1+\alpha}} \text{ as } n \to \infty.$$

Proof of Lemma 2. We split the analysis into two cases:
Case 1. $\alpha < 0$. Setting $x := -\alpha$, we have that

$$\left|\binom{\alpha}{n}\right| = \left|\binom{-x}{n}\right| = \left|\frac{-x(-x-1)\dots(-x-n+1)}{n!}\right| = \frac{x(x+1)\dots(x+n-1)(x+n)}{n!(x+n)},$$

and so

$$\lim_{n\to\infty} n^{1-x}\left|\binom{-x}{n}\right| = \lim_{n\to\infty} \frac{x(x+1)\dots(x+n)}{n!n^x}\frac{n}{n+x} = \frac{1}{\Gamma(x)}.$$

The computation of this last limit requires some justification which will be provided in the course of the solution to Problem 4(c) below, see (7). Assuming it for the time being, we get the desired result with $c(\alpha) := \Gamma(-\alpha)^{-1}$.

Case 2. $k-1 < \alpha < k$ for some natural number $k \geq 1$. In this case we can apply Case 1 to $\alpha' = \alpha - k < 0$ to get that

$$\lim_{n\to\infty}\left|\binom{\alpha-k}{n}\right| n^{1+\alpha-k} = \frac{1}{\Gamma(k-\alpha)} \text{ and } \lim_{n\to\infty}\left|\binom{\alpha-k}{n-k}\right| n^{1+\alpha-k} = \frac{1}{\Gamma(k-\alpha)}.$$

Since

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{n(n-1)\dots(n-k+1)}\binom{\alpha-k}{n-k},$$

we conclude that

$$\begin{aligned}\lim_{n\to\infty}\left|\binom{\alpha}{n}\right| n^{1+\alpha} &= \lim_{n\to\infty}\frac{n^k\alpha(\alpha-1)\dots(\alpha-k+1)}{n(n-1)\dots(n-k+1)}\left|\binom{\alpha-k}{n-k}\right| n^{1+\alpha-k} \\ &= \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{\Gamma(k-\alpha)} =: c(\alpha).\end{aligned}$$ □

(b1) By a previous remark, we lose no generality in assuming that $\alpha \in (0,\infty) \setminus \mathbb{N}$. By Lemma 2, there exists a constant $C > 0$ such that

$$\left|\binom{\alpha}{n}\right| \leq \frac{C}{n^{1+\alpha}} \text{ for every } n \geq 1.$$

Since the series

$$\sum_{n=1}^{\infty}\frac{1}{n^{1+\alpha}} < \infty$$

converges for $\alpha > 0$, the result follows.

(b2) For $-1 < \alpha < 0$ one has that

$$\binom{\alpha}{n} = (-1)^n\left|\binom{\alpha}{n}\right|.$$

The convergence of the binomial series at $x = 1$ follows from the Leibniz criterion for alternating series since Lemma 2 implies that

$$\lim_{n\to\infty}\left|\binom{\alpha}{n}\right| = 0 \text{ if } \alpha > -1.$$

The divergence of the binomial series at $x = -1$ follows from the fact that the series

$$\sum_{n=1}^{\infty}\frac{1}{n^{1+\alpha}} = \infty$$

diverges if $\alpha < 0$.

[2] Given two positive sequences $\{a_n\}$ and $\{b_n\}$, we write $a_n \sim b_n$ to mean that

$$\lim_{n\to\infty}\frac{a_n}{b_n} = 1.$$

(b3) From Lemma 2 it follows that

$$\lim_{n\to\infty}\left|\binom{\alpha}{n}\right| \neq 0 \text{ if } \alpha \leq -1.$$

As a consequence, if $\alpha \leq -1$, both of the following series diverge:

$$\sum_{n=0}^{\infty}\binom{\alpha}{n} \text{ and } \sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n.$$

Problem 4 (Beta function)

(a) Start by noting that the integral defining the Beta function is only improper if $0 < x < 1$ or $0 < y < 1$, for otherwise we are integrating a continuous function over a bounded interval.

Let us assume without loss of generality that the first case holds, i.e. $0 < x < 1$, and show that the limit

$$\lim_{\epsilon\searrow 0^+}\int_{\epsilon}^{1/2} t^{x-1}(1-t)^{y-1}dt$$

exists as a real number. The function $t \mapsto (1-t)^{y-1}$ is continuous on the closed interval $[0, 1/2]$ and therefore bounded. As a consequence, there exists $C \in (0,\infty)$ for which

$$|t^{x-1}(1-t)^{y-1}| \leq Ct^{x-1}, \text{ for every } 0 \leq t \leq \frac{1}{2},$$

and the result follows from the convergence of the improper integral

$$\int_0^{1/2} t^{x-1}dt$$

for $x > 0$ (which was established in Problem 3(b), ÜB 12).

(b) Let us start by establishing the functional equation

$$x\mathrm{B}(x,y) = (x+y)\mathrm{B}(x+1,y). \tag{3}$$

Integrating by parts,[3]

$$x\mathrm{B}(x,y+1) = \int_0^1 xt^{x-1}(1-t)^y dt = y\int_0^1 t^x(1-t)^{y-1}dt = y\mathrm{B}(x+1,y).$$

On the other hand,

$$\begin{aligned}\mathrm{B}(x,y+1) &= \int_0^1 t^{x-1}(1-t)(1-t)^{y-1}dt\\ &= \int_0^1 t^{x-1}(1-t)^{y-1}dt - \int_0^1 t^x(1-t)^{y-1}dt\\ &= \mathrm{B}(x,y) - \mathrm{B}(x+1,y).\end{aligned}$$

These two identities together yield (3).

Now, let us fix $y > 0$ and prove that

$$\mathrm{B}(\lambda x_1 + (1-\lambda)x_2, y) \leq \mathrm{B}(x_1,y)^{\lambda}\mathrm{B}(x_2,y)^{1-\lambda}$$

for every $x_1, x_2 > 0$ and $0 < \lambda < 1$. As with the Gamma function (see the solution of Problem 4(c) from ÜB 12), our main tool will be Hölder's inequality. To set the stage, define p, q via $\lambda =: p^{-1}$ and $1-\lambda =: q^{-1}$. If

$$f(t) := t^{(x_1-1)/p}(1-t)^{(y-1)/p} \text{ and } g(t) := t^{(x_2-1)/q}(1-t)^{(y-1)/q},$$

then

$$f(t)g(t) = t^{\lambda x_1+(1-\lambda)x_2-1}(1-t)^{y-1}.$$

We conclude that

$$\mathrm{B}(\lambda x_1+(1-\lambda)x_2,y) = \int_0^1 f(t)g(t)dt \leq \Big(\int_0^1 f(t)^p dt\Big)^{1/p}\Big(\int_0^1 g(t)^q dt\Big)^{1/q} = \mathrm{B}(x_1,y)^{\lambda}\mathrm{B}(x_2,y)^{1-\lambda},$$

as desired.

[3] Notice that the boundary terms vanish since the function $t \mapsto t^x(1-t)^y$ vanishes at both endpoints $t = 0$ and $t = 1$.

(c) We make use of Problem 4 from PÜ 11, formulating it as a theorem and proving it below:

Theorem 1. Let $F : (0, \infty) \to (0, \infty)$ be a function satisfying the following conditions:

(i) $F(1) = 1$;

(ii) $F(x + 1) = xF(x)$ for every $0 < x < \infty$;

(iii) F is logarithmically convex.

Then $F(x) = \Gamma(x)$ for every $0 < x < \infty$.

Proof of Theorem 1. Since the Gamma function Γ does satisfy conditions $(i) - (iii)$ above (see Problem 4, ÜB 12), it will be enough to show that a function satisfying these conditions is uniquely determined. From (ii) it follows that

$$F(x + n) = x(x + 1) \dots (x + n - 1)F(x) \tag{4}$$

for every real $x > 0$ and natural $n \geq 1$. In particular, $F(n + 1) = n!$ for every $n \in \mathbb{N}$, and so it suffices to show that $F(x)$ is uniquely determined on the interval $0 < x < 1$. Since

$$n + x = (1 - x)n + x(n + 1),$$

the logarithmic convexity and the functional equation for F imply that

$$F(n + x) \leq F(n)^{1-x}F(n + 1)^x = F(n)^{1-x}(nF(n))^x = (n - 1)!n^x. \tag{5}$$

On the other hand, the convex combination

$$n + 1 = x(n + x) + (1 - x)(n + x + 1)$$

implies likewise that

$$n! = F(n + 1) \leq F(n + x)^x F(n + x + 1)^{1-x} = F(n + x)(n + x)^{1-x}. \tag{6}$$

Inequalities (5) and (6) can be combined to yield

$$n!(n + x)^{x-1} \leq F(n + x) \leq (n - 1)!n^x,$$

and so from (4) it follows that

$$a_n(x) := \frac{n!(n + x)^{x-1}}{x(x + 1) \dots (x + n - 1)} \leq F(x) \leq \frac{(n - 1)!n^x}{x(x + 1) \dots (x + n - 1)} =: b_n(x).$$

Since

$$\lim_{n\to\infty} \frac{b_n(x)}{a_n(x)} = \lim_{n\to\infty} \frac{(n + x)n^x}{n(n + x)^x} = 1,$$

it follows by squeezing that

$$F(x) = \lim_{n\to\infty} \frac{(n - 1)!n^x}{x(x + 1) \dots (x + n - 1)}.$$

We conclude that F is uniquely determined, as desired. □

Remark. As a consequence of the proof of Theorem 1, we have the following representation for the Gamma function:

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x + 1) \dots (x + n)} \quad \text{for every } x > 0. \tag{7}$$

Back to the solution of the original problem: let $y > 0$ be fixed and define the function $F : (0, \infty) \to (0, \infty)$ via

$$F(x) := \mathrm{B}(x, y)\frac{\Gamma(x + y)}{\Gamma(y)}.$$

The plan is to show that the function F satisfies conditions $(i), (ii)$ and (iii) of Theorem 1. As a consequence, $F(x) = \Gamma(x)$ for every real $x > 0$, and the result follows.

Start by noting that

$$F(1) = \mathrm{B}(1, y)\frac{\Gamma(1 + y)}{\Gamma(y)} = y\mathrm{B}(1, y) = y \int_0^1 (1 - t)^{y-1}dt = -(1 - t)^y\Big|_0^1 = 1,$$

and so condition (i) is fulfilled. From (3) and the functional equation for the Gamma function, we also have that

$$F(x+1) = \mathrm{B}(x+1,y)\frac{\Gamma(x+y+1)}{\Gamma(y)} = \frac{x}{\cancel{x+y}}\mathrm{B}(x,y)\frac{\cancel{(x+y)}\Gamma(x+y)}{\Gamma(y)} = xF(x),$$

which proves (ii). Finally, F is logarithmically convex because it is the product of (a constant $\Gamma(y)^{-1}$ times) two logarithmically convex functions, namely $x \mapsto \mathrm{B}(x,y)$ and $x \mapsto \Gamma(x+y)$. This establishes (iii) and concludes the proof.

We finish with a sketch of the function $(x,y) \mapsto \mathrm{B}(x,y)$ on the region $(0,1) \times (0,1)$: