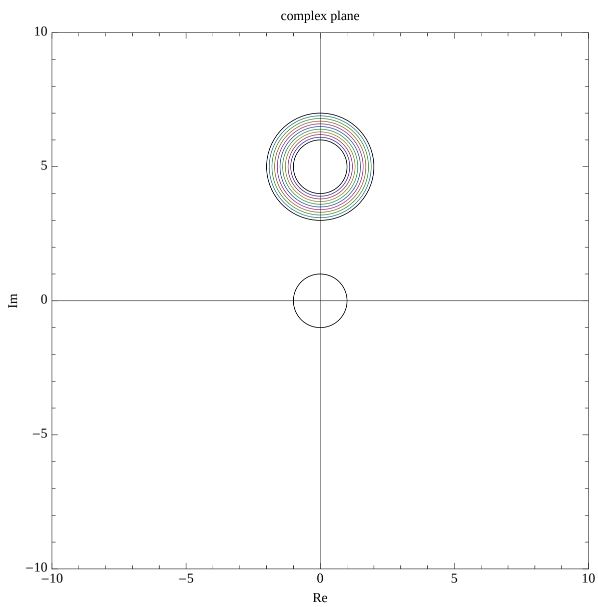
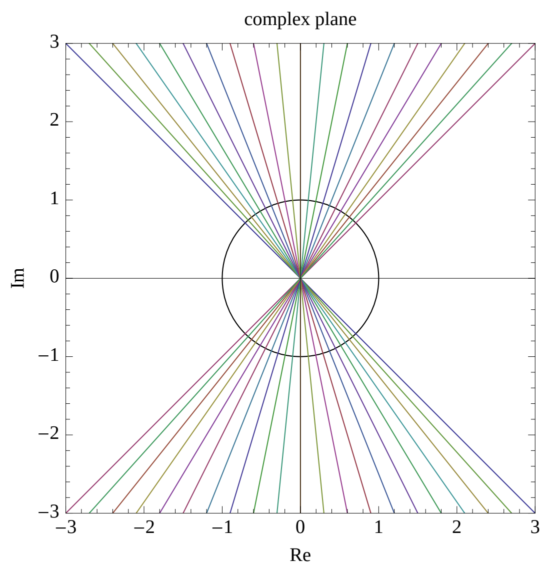
Problem 1 (Complex numbers)

(a) The set A consists of the *open* annulus centered at the point $5i$ of inner radius 1 and outer radius 2, as depicted (together with the unit circle) in the following picture:



To determine the set B, start by noting that, if $z = x + yi$, then $z^2 = (x^2 - y^2) + (2xy)i$. It follows that the condition $\Re(z^2) < 0$ is equivalent to $x^2 - y^2 < 0$, which in turns holds if and only if $|x| < |y|$. Thus B coincides with the *interior* of the colored region depicted in the following picture:



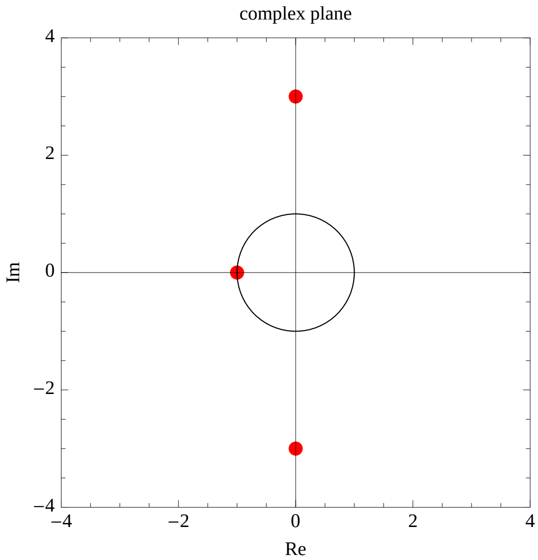
By the fundamental theorem of algebra, the set C consists of at most 3 points. We claim that C consists of exactly 3 points, which in turn can be determined by noting that $z_1 := -1$ is a root of the polynomial

$$p(z) := z^3 + z^2 + 9z + 9.$$

By polynomial division,

$$\frac{p(z)}{z+1} = 9 + z^2,$$

and so the other two points z_2, z_3 are solutions of the quadratic equation $9 + z^2 = 0$, and are thus given by $z_2 = 3i$ and $z_3 = -3i$. The three points $\{z_1, z_2, z_3\} = \{-1, 3i, -3i\}$ are represented in the following picture:



(b) The reader whose memory has survived the Christmas break will recognize that all the work has already been done in ÜB11, Problem 1(b). Namely, the identity

$$|1 - \overline{w}z|^2 - |w - z|^2 = (1 - |w|^2)(1 - |z|^2)$$

provides a quantitative version for the qualitative statement which we are trying to prove. Instead of finishing it here, let us instead offer an alternative solution, and sketch how yet another (higher-level) approach could be implemented:

Alternative solution. Let $w \in \mathbb{C}$ with $|w| < 1$ be given. Given $z \in \mathbb{C}$, there exist $r \geq 0$ and $\theta \in [0, 2\pi)$ such that $z = re^{i\theta}$. Having fixed θ, we can now write $w = \omega e^{i\theta}$ for some complex number ω *of the same modulus as* w. In particular, ω still lies inside the unit disc. It follows that

$$\left|\frac{z - w}{1 - \overline{w}z}\right| = \left|\frac{re^{i\theta} - \omega e^{i\theta}}{1 - \overline{\omega}\cancel{e^{-i\theta}}r\cancel{e^{i\theta}}}\right| = \left|\frac{e^{i\theta}(r - \omega)}{1 - \overline{\omega}r}\right| = \left|\frac{r - \omega}{1 - \overline{\omega}r}\right|.$$

Therefore, if $|z| \leq 1$, we lose no generality in assuming that $z = r$ is a real number such that $0 \leq r \leq 1$. We want to show that $|r - \omega| \leq |1 - \overline{\omega}r|$, which happens if and only if

$$(r - \omega)(r - \overline{\omega}) \leq (1 - \overline{\omega r})(1 - \omega r).$$

Expanding both sides, making all the necessary cancellations and moving all the terms to the right-hand side, we see that this inequality is equivalent to

$$0 \leq (1 - r^2)(1 - |\omega|^2),$$

which holds since $r = |z| \leq 1$ and $|\omega| = |w| < 1$. Incidentally, this concludes the exercise since the same argument applied to any $|z| > 1$ leads to the reversed inequality

$$0 \geq (1 - r^2)(1 - |\omega|^2).$$

Sketch of another alternative solution. Given $w \in \mathbb{C}$ inside the open unit disc, define a map

$$\begin{array}{rcl} \varphi_w : \mathbb{C} & \to & \mathbb{C} \\ z & \mapsto & \frac{z - w}{1 - \overline{w}z}. \end{array}$$

It is easy to check that φ_w is a continuous map that carries the unit circle onto itself and the origin into a complex number of modulus less than 1 (namely, $-w$). Moreover, φ_w is a bijection whose inverse is given by $\varphi_w^{-1} = \varphi_{-w}$. The result now follows from the maximum modulus principle of complex analysis, or from the more elementary fact (which still requires a short proof) that the continuous image of a (path-)connected set is still (path-)connected.

Maps like the ones given by the family $\{\varphi_w\}$ are examples of the so-called *fractional linear transformations* (or Möbius transformations) and constitute an important starting point of the study of several more advanced areas of mathematics.

Problem 2 (Euler-Mascheroni)

Start by noting that, for any $k \in \mathbb{N} \setminus \{0\}$,

$$\int_{\frac{1}{k+1}}^{\frac{1}{k}} f(x)dx = \int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{1}{x}dx - \int_{\frac{1}{k+1}}^{\frac{1}{k}} \left\lfloor \frac{1}{x} \right\rfloor dx.$$

Now, if $\frac{1}{k+1} < x \leq \frac{1}{k}$, then $k \leq \frac{1}{x} < k+1$, and so $\lfloor \frac{1}{x} \rfloor = k$. It follows that

$$\begin{aligned}
\int_{\frac{1}{k+1}}^{\frac{1}{k}} f(x)dx &= \int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{1}{x}dx - \int_{\frac{1}{k+1}}^{\frac{1}{k}} kdx \\
&= \ln\Big(\frac{1}{k}\Big) - \ln\Big(\frac{1}{k+1}\Big) - k\Big(\frac{1}{k} - \frac{1}{k+1}\Big) \\
&= \ln\Big(\frac{1}{k}\Big) - \ln\Big(\frac{1}{k+1}\Big) - \frac{1}{k+1}.
\end{aligned}$$

The interval $(0,1]$ can be partitioned in the following way:

$$(0,1] = \bigcup_{k=1}^{\infty} \Big(\frac{1}{k+1}, \frac{1}{k}\Big]$$

It follows that

$$\begin{aligned}
\lim_{\epsilon \searrow 0^+} \int_{\epsilon}^{1} f(x)dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \int_{\frac{1}{k+1}}^{\frac{1}{k}} f(x)dx \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \Big\{ \ln\Big(\frac{1}{k}\Big) - \ln\Big(\frac{1}{k+1}\Big) - \frac{1}{k+1} \Big\} \\
&= \lim_{n\to\infty} \Big(- \ln\Big(\frac{1}{n+1}\Big) - \sum_{k=1}^{n} \frac{1}{k+1} \Big) \\
&= \lim_{n\to\infty} \Big(\ln(n) - \sum_{k=2}^{n} \frac{1}{k} \Big).
\end{aligned}$$

We will be done once we show that this limit exists (as a finite real number). With this goal in mind, compare the integral

$$\int_1^n \frac{dx}{x} = \ln(n)$$

with its lower and upper Riemann sums to conclude that

$$\sum_{k=2}^{n} \frac{1}{k} \leq \ln(n) \leq \sum_{k=1}^{n-1} \frac{1}{k}.$$

It follows that

$$\gamma_n := \sum_{k=1}^{n} \frac{1}{k} - \ln(n)$$

satisfies

$$\frac{1}{n} \leq \gamma_n \leq 1,$$

and that the difference of two consecutive elements of the sequence $\{\gamma_n\}$ satisfies

$$\gamma_{n-1} - \gamma_n = \ln(n) - \ln(n-1) - \frac{1}{n} = \int_{n-1}^{n} \frac{dx}{x} - \frac{1}{n} = \int_{n-1}^{n} \Big(\frac{1}{x} - \frac{1}{n}\Big)dx > 0.$$

The sequence $\{\gamma_n\}$ is therefore monotonically decreasing. Since it is bounded from below by 0, it converges to a finite real number, say γ. It can be shown that

$$\gamma := \lim_{n\to\infty} \gamma_n \simeq 0.57721566490153286061.$$

Going back to the original question, we finally conclude that

$$\lim_{\epsilon\searrow 0^+}\int_\epsilon^1 f(x)dx = \lim_{n\to\infty}\left(\ln(n) - \sum_{k=2}^n \frac{1}{k}\right) = 1-\gamma \simeq 0.42278.$$

Remark. The constant γ is known in the literature as the *Euler-Mascheroni constant.* Despite its multiple appearances in different branches of mathematics, it remains a rather mysterious number. For instance, it is not even known whether γ is irrational or not!

Problem 3 (Improper integrals)

(a) The integral $\int_1^\infty \frac{dx}{x^s}$ converges if $s>1$. In this case, note that the function $F(x) = \frac{x^{-s+1}}{-s+1}$ is such that

$$F'(x) = \frac{1}{x^s},$$

and compute: given a large $R>0$,

$$\int_1^R \frac{dx}{x^s} = \int_1^R F'(x)dx = F(R) - F(1) = \frac{R^{-s+1}}{-s+1} - \frac{1^{-s+1}}{-s+1}.$$

Since $\lim_{R\to\infty} R^{-s+1} = 0$, it follows that

$$\int_1^\infty \frac{dx}{x^s} = \frac{1}{s-1}. \quad (s>1)$$

On the other hand, if $s\le 1$, then the integral $\int_1^\infty \frac{dx}{x^s}$ does *not* converge. Perhaps the easiest way to see this is to note that, for $s=1$,

$$\int_1^R \frac{dx}{x} = \ln(R),$$

which blows up as $R\to\infty$. By comparison, it then follows that, for every $s<1$,

$$\int_1^\infty \frac{dx}{x^s} \ge \int_1^\infty \frac{dx}{x} = \infty$$

diverges as well.

(b) The situation becomes reversed with respect to part (a) in terms of the ranges of s for which the integral converges/diverges. In more detail: if $s<1$, choose a small $\epsilon>0$ and note that

$$\int_\epsilon^1 \frac{dx}{x^s} = \frac{1}{1-s}\frac{1}{x^{s-1}}\Big|_\epsilon^1 = \frac{1}{1-s}(1-\epsilon^{1-s}).$$

Since $\lim_{\epsilon\searrow 0^+}\epsilon^{1-s} = 0$, it follows that

$$\int_0^1 \frac{dx}{x^s} = \frac{1}{1-s}. \quad (s<1)$$

On the other hand,

$$\int_0^1 \frac{dx}{x} = \lim_{\epsilon\searrow 0^+} \ln(x)|_\epsilon^1 = \ln(1) - \ln(0) = \infty$$

diverges, and so does $\int_0^1 \frac{dx}{x^s}$ for any $s>1$.

Problem 4 (Gamma function)

(a) Let $x>0$. The strategy will be to break up the integral defining $\Gamma(x)$ into two pieces,

$$\int_0^\infty t^{x-1}e^{-t}dt = \int_0^{t_0} t^{x-1}e^{-t}dt + \int_{t_0}^\infty t^{x-1}e^{-t}dt,$$

($t_0<\infty$ will be chosen below) and to estimate each piece separately.

For the first piece, observe that $e^{-t}\le 1$ if $0\le t\le t_0$, and so

$$\int_0^{t_0} t^{x-1}e^{-t}dt \le \int_0^{t_0} t^{x-1}dt = \frac{t^x}{x}\Big|_0^{t_0} = \frac{t_0^x}{x} < \infty.$$

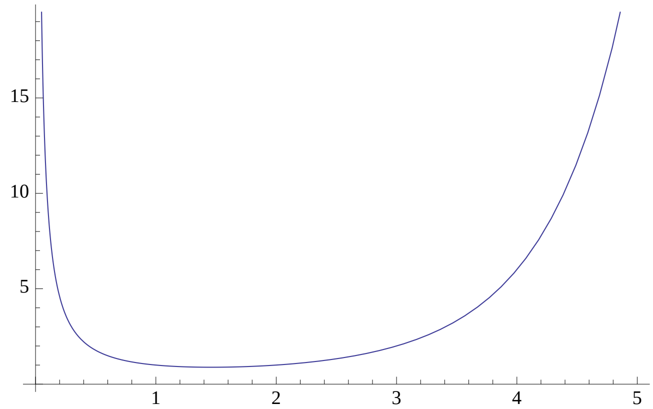
Let $t_0 > 0$ be chosen large enough such that

$$t^{x-1}e^{-t} \leq \frac{1}{t^2} \text{ if } t \geq t_0.$$

Then the second piece can be easily estimated as follows:

$$\int_{t_0}^{\infty} t^{x-1}e^{-t}dt \leq \int_{t_0}^{\infty} \frac{dt}{t^2} = \frac{1}{t_0} < \infty.$$

The result follows. Below is a picture of the function Γ on the positive real axis:



(b) Let us start by establishing the functional equation

$$x\Gamma(x) = \Gamma(x+1), \text{ for every } x > 0. \tag{1}$$

Integrating by parts, we have that

$$x\Gamma(x) = \int_0^\infty (xt^{x-1})e^{-t}dt = \int_0^\infty (t^x)'e^{-t}dt \overset{(!)}{=} -\int_0^\infty t^x(e^{-t})'dt = \int_0^\infty t^x e^{-t}dt = \Gamma(x+1).$$

Notice that no boundary terms are picked up in (!) because of the vanishing of the function $t \mapsto t^x e^{-t}$ at $t = 0$ and at $t = \infty$. This establishes (1).

As a consequence,

$$n\Gamma(n) = \Gamma(n+1) \text{ for every } n \in \mathbb{N} \setminus \{0\}. \tag{2}$$

The claimed formula $\Gamma(n+1) = n!$, valid for every $n \in \mathbb{N}$, follows from (2) by induction, if one just realizes that

$$\Gamma(1) = \int_0^\infty e^{-t}dt = -e^{-t}|_0^\infty = 0 - (-1) = 1.$$

Thus Γ is a continuous extension of the factorial sequence to the whole interval $(0, \infty)$.

(c) Start by noting that $\Gamma(x) > 0$ for every $x > 0$. This follows from the integral definition of Γ.

Now, let $x, y \in (0, \infty)$ and $0 < \lambda < 1$. The goal is to show that

$$\ln\Gamma(\lambda x + (1-\lambda)y) \leq \lambda \ln\Gamma(x) + (1-\lambda)\ln\Gamma(y). \tag{3}$$

Set $p = \frac{1}{\lambda}$ and $q = \frac{1}{1-\lambda}$, so that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

It suffices to show that

$$\Gamma\Big(\frac{x}{p} + \frac{y}{q}\Big) \leq \Gamma(x)^{\frac{1}{p}}\Gamma(y)^{\frac{1}{q}}, \tag{4}$$

for then the desired inequality (3) follows by taking logarithms on both sides:

$$\ln\Gamma(\lambda x+(1-\lambda)y) = \ln\Big(\Gamma\Big(\frac{x}{p}+\frac{y}{q}\Big)\Big) \leq \ln\Big(\Gamma(x)^{\frac{1}{p}}\Gamma(y)^{\frac{1}{q}}\Big) = \frac{1}{p}\ln\Gamma(x)+\frac{1}{q}\ln\Gamma(y) = \lambda\ln\Gamma(x)+(1-\lambda)\ln\Gamma(y).$$

To verify (4), consider the auxiliary functions

$$f(t) := t^{\frac{x-1}{p}}e^{-\frac{t}{p}} \text{ and } g(t) := t^{\frac{y-1}{q}}e^{-\frac{t}{q}},$$

which satisfy

$$f(t)g(t) = t^{\frac{x}{p}+\frac{y}{q}-1}e^{-t}, \quad f(t)^p = t^{x-1}e^{-t}, \text{ and } g(t)^q = t^{y-1}e^{-t}. \tag{5}$$

Choose a small $\epsilon > 0$ and a large $R < \infty$. Then Hölder's inequality on the bounded interval $[\epsilon, R]$ implies that

$$\int_\epsilon^R f(t)g(t)dt \leq \Big(\int_\epsilon^R f(t)^p dt\Big)^{\frac{1}{p}} \Big(\int_\epsilon^R g(t)^q dt\Big)^{\frac{1}{q}}.$$

Letting $\epsilon \searrow 0^+$ and $R \nearrow \infty$, we have that

$$\int_0^\infty f(t)g(t)dt \leq \Big(\int_0^\infty f(t)^p dt\Big)^{\frac{1}{p}} \Big(\int_0^\infty g(t)^q dt\Big)^{\frac{1}{q}},$$

which in light of (5) translates into

$$\int_0^\infty t^{\frac{x}{p}+\frac{y}{q}-1}e^{-t}dt \leq \Big(\int_0^\infty t^{x-1}e^{-t}dt\Big)^{\frac{1}{p}} \Big(\int_0^\infty t^{y-1}e^{-t}dt\Big)^{\frac{1}{q}}.$$

Recalling the integral definition of Γ, we see that is exactly our desired estimate (4). The proof is complete, and we complement it with the following illustration of the function $\ln\Gamma$: