Analysis 1, Solutions to problem set 10

Problem 1 (Continuity)

(a) Let us start by verifying that the function f is continuous at $x = 0$. We make use of Definition 5.27, by which it is enough to show that

$$\forall \epsilon > 0 \exists \delta > 0 \forall y \in [0, 1] : |0 - y| < \delta \Rightarrow |f(0) - f(y)| < \epsilon.$$

Let $\epsilon > 0$ be given, and choose $\delta = \epsilon$. Let y be any number in $[0, 1]$ such that

$$|0 - y| = |y| < \epsilon.$$

We split the analysis into two cases, depending on whether y is a dyadic number or not. In the first case, $y \in \mathbb{Y}$, we have that

$$|f(0) - f(y)| = |0 - f(y)| = |f(y)| = |y| < \epsilon.$$

In the second case, $y \in [0, 1] \setminus \mathbb{Y}$, we have that

$$|f(0) - f(y)| = |0 - 0| = 0 < \epsilon.$$

In both cases we conclude that

$$|f(0) - f(y)| < \epsilon.$$

It follows that the function f is continuous at 0, as desired.

Let $x_0 \in [0, 1] \setminus \{0\} =: (0, 1]$ be given. Our next goal is to show that the function f is discontinuous at $x = x_0$. Again by Definition 5.27, this amounts to showing

$$\exists \epsilon > 0 \forall \delta > 0 \exists x \in [0, 1] : |x_0 - x| < \delta \wedge |f(x_0) - f(x)| \geq \epsilon.$$

We claim that taking $\epsilon = x_0/2$ will do the job. (Notice that this is a positive number *since* $x_0 \neq 0$.) Again we split the analysis into two cases, according to whether x_0 is a dyadic number or not.

In the first case, $x_0 \in \mathbb{Y}$, let $\epsilon := x_0/2$ and let $\delta > 0$ be given. Pick $x \in [0, 1] \setminus \mathbb{Y}$ such that

$$|x_0 - x| < \delta.$$

That such x exists can be seen in a number of ways (e.g. by using the fact that $\{\sqrt{2}/n\}_{n \geq 2}$ is a sequence of elements in $[0, 1] \setminus \mathbb{Y}$ converging to 0). Since $x_0 \in \mathbb{Y}$, we have that $f(x_0) = x_0$. Since $x \in [0, 1] \setminus \mathbb{Y}$, we have that $f(x) = 0$. But then

$$|f(x_0) - f(x)| = |x_0 - 0| = x_0 \geq \frac{x_0}{2} = \epsilon.$$

In the second case, $x_0 \in [0, 1] \setminus \mathbb{Y}$, let again $\epsilon := x_0/2$ and let $\delta > 0$ be given. We lose no generality in assuming that $\delta < x_0/2$, for if $|x - x_0| < \delta$ holds for some $\delta = \delta_0$, it will also hold for any $\delta > \delta_0$. Pick $x \in \mathbb{Y} \cap [0, 1]$ such that

$$|x - x_0| < \delta.$$

That such x exists is an immediate consequence of Theorem 2.18 (Archimedes). Reasoning similarly as before, we conclude that

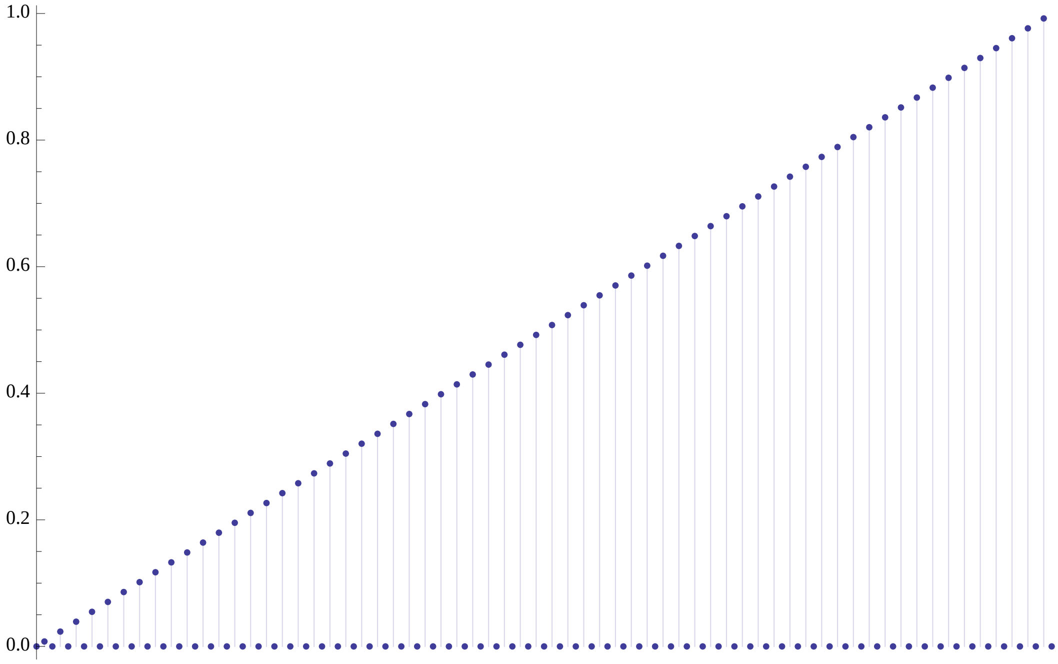
$$|f(x_0) - f(x)| = |0 - x| = x > x_0 - \delta \geq \frac{x_0}{2} = \epsilon.$$

In both cases, we were able to produce an absolute $\epsilon > 0$ and a number $x \in [0, 1]$ (depending on δ but on nothing else) such that

$$|x_0 - x| < \delta \text{ but } |f(x_0) - f(x)| \geq \epsilon.$$

This means that the function f is discontinuous at $x = x_0$, as claimed.

An approximate graphical representation of the function f is as follows (in this particular representation, points in $[0, 1] \cap \bigcup_{k=0}^{6} \mathbb{Y}_k$ appear with the correct f-value):



(b) Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f([a, b]) \subset [a, b]$. We want to show that $f(x_0) = x_0$ for some $x_0 \in [a, b]$.

We lose no generality in assuming that $f(a) \neq a$ and $f(b) \neq b$, for otherwise we would be done. If $f(a) \neq a$ and $f([a, b]) \subset [a, b]$, then $f(a) > a$, or $a - f(a) < 0$. Similarly, $f(b) < b$, or $b - f(b) > 0$. Consider the auxiliary function

$$\begin{aligned} g : [a, b] &\to \mathbb{R} \\ x &\mapsto g(x) := x - f(x). \end{aligned}$$

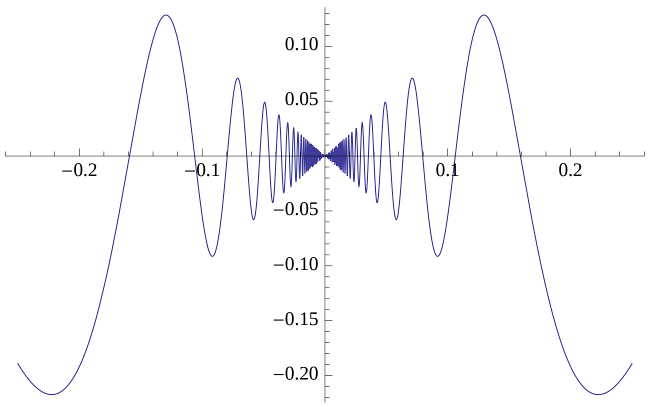
Then g is a continuous function (since it is the sum of two continuous functions) which satisfies

$$\begin{aligned} g(a) &= a - f(a) < 0, \text{ and} \\ g(b) &= b - f(b) > 0. \end{aligned}$$

A straightforward modification of the Intermediate Value Theorem (Theorem[1] 5.15 from the Skript) implies that the function g must have a zero in the interval (a, b). In other words, there exists $x_0 \in (a, b)$ such that $g(x_0) = 0$. This happens if and only if $f(x_0) = x_0$ i.e. if and only if x_0 is a fixed point for f, and we are done.

Problem 2 (Differentiability)

(a) Here's a sketch of the function f_1 on the interval $[-1/4, 1/4]$:



We want to show that the function f_1 is *not* differentiable at $x = 0$. This amounts to showing that the limit

$$\lim_{h \to 0} \frac{f_1(0 + h) - f_1(0)}{h} = \lim_{h \to 0} \frac{f_1(h) - 0}{h} = \lim_{h \to 0} \frac{1}{\cancel{h}} \cancel{h} \sin\left(\frac{1}{h}\right) = \lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$

[1]Note carefully that the monotonicity assumption can be dropped.

does not exist. This might already have been discussed in Problem 4(a) from PB10, but we recall it here. Define the sequences

$$a_n := \frac{1}{\pi n} \text{ and } b_n := \frac{1}{\frac{\pi}{2} + \pi n}.$$

Then

$$\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n,$$

but

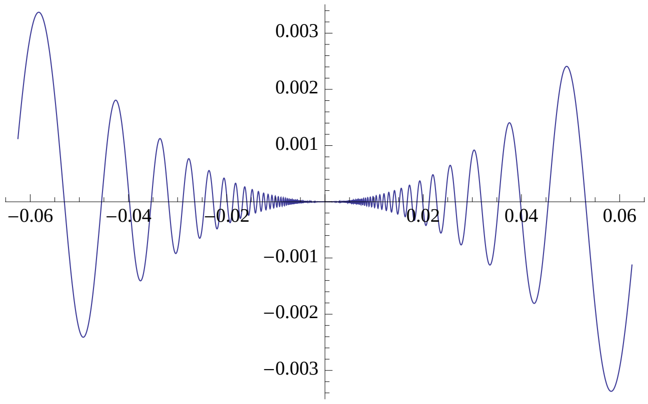
$$\lim_{n\to\infty} \sin\left(\frac{1}{a_n}\right) = \lim_{n\to\infty} \sin(\pi n) = 0 \neq 1 = \limsup_{n\to\infty} \left\{ \sin\left(\frac{\pi}{2} + \pi n\right)\right\} = \limsup_{n\to\infty} \left\{ \sin\left(\frac{1}{b_n}\right)\right\}.$$

This shows that the limit

$$\lim_{h\to 0} \sin\left(\frac{1}{h}\right)$$

does not exist, and therefore the function f_1 is not differentiable at 0, as desired.

(b) Here's a sketch of the function f_2 on the interval $[-1/16, 1/16]$:



This time we need to show that the limit

$$\lim_{h\to 0} \frac{f_2(0+h) - f_2(0)}{h} = \lim_{h\to 0} \frac{f_2(h) - 0}{h} = \lim_{h\to 0} \frac{1}{h} h^2 \sin\left(\frac{1}{h}\right) = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right)$$

exists and is a finite real number. Indeed, we claim that

$$\lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0, \tag{1}$$

and this is the content of Problem 4(b) from PB10. We recall it here: first of all,

$$\lim_{h\to 0^+} h \sin\left(\frac{1}{h}\right) = \lim_{h\to 0^+} \frac{\sin\left(\frac{1}{h}\right)}{\frac{1}{h}} = \lim_{t\to +\infty} \frac{\sin t}{t}.$$

In the solution of Problem 1(a) from ÜB9, we have already argued that $\lim_{t\to+\infty} \frac{\sin t}{t} = 0$, and so

$$\lim_{h\to 0^+} h \sin\left(\frac{1}{h}\right) = 0.$$

Similarly,

$$\lim_{h\to 0^-} h \sin\left(\frac{1}{h}\right) = 0.$$

Since the two side limits equal zero, (1) follows. Moreover,

$$f_2'(0) = \lim_{h\to 0} \frac{f_2(0+h) - f_2(0)}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0.$$

(c) From part (b) we already know that $f_2'(0) = 0$. If $x \neq 0$, then the product rule implies that

$$f_2'(x) = \left(x^2 \sin\left(\frac{1}{x}\right)\right)' = (x^2)' \sin\left(\frac{1}{x}\right) + x^2 \left(\sin\left(\frac{1}{x}\right)\right)'.$$

We have seen in class that $(x^2)' = 2x$. On the other hand, it is easy to check that $(x^{-1})' = -x^{-2}$, and then the chain rule implies that

$$\left(\sin\left(\frac{1}{x}\right)\right)' = \left(\frac{1}{x}\right)' \sin'\left(\frac{1}{x}\right) = -\frac{1}{x^2} \cos\left(\frac{1}{x}\right)$$

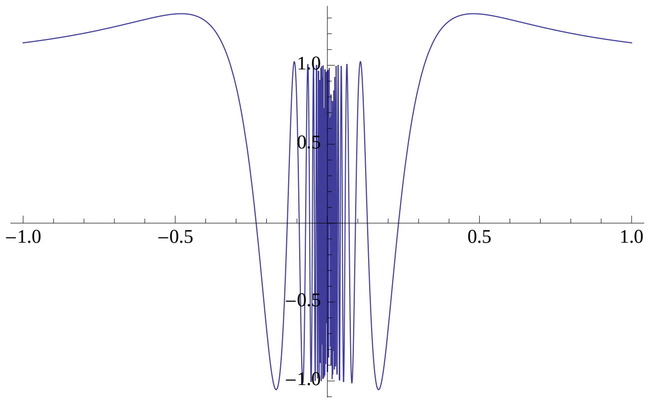
since $\sin' = \cos$. It follows that

$$\begin{aligned} f_2'(x) &= (x^2)' \sin\left(\frac{1}{x}\right) + x^2 \Big(\sin\left(\frac{1}{x}\right)\Big)' \\ &= 2x \sin\left(\frac{1}{x}\right) + x^2 \Big(-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\Big) \\ &= 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right). \end{aligned}$$

We conclude that

$$f_2'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \in (-1,1) \setminus \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

This is a rather pathological function (for instance, it is not continuous at $x = 0$) which is nevertheless bounded on the interval $(-1, 1)$ and looks approximately as follows:



Problem 3 (A differential inequality)

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying

$$f'(x) > f(x), \ \forall x \in \mathbb{R}. \tag{2}$$

(a) Let $x_0 \in \mathbb{R}$ be such that $f(x_0) = 0$. By definition,

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

Since $f(x_0) = 0$, assumption (2) at $x = x_0$ translates into $f'(x_0) > 0$, or

$$\lim_{h\to 0} \frac{f(x_0+h) - 0}{h} > 0.$$

In particular,

$$\lim_{h\to 0^+} \frac{f(x_0+h)}{h} > 0.$$

We can therefore guarantee the existence of $h_0 > 0$ such that

$$\frac{f(x_0+h)}{h} > 0$$

provided $0 < h < h_0$. Pick such h. Since $h > 0$, this implies that

$$f(x_0+h) > 0.$$

Set $x := x_0 + h$. Then $x > x_0$ because $h > 0$, and $f(x) = f(x_0 + h) > 0$, and we are done.

(b) Since the exponential function is (strictly) positive on the whole real line, assumption (2) is equivalent to the following statement:

$$e^{-x} f'(x) > e^{-x} f(x), \forall x \in \mathbb{R}.$$

This can be rewritten as

$$e^{-x} f'(x) - e^{-x} f(x) > 0, \forall x \in \mathbb{R},$$

or even

$$(e^{-x}f(x))' > 0, \forall x \in \mathbb{R}. \tag{3}$$

This last reformulation is a consequence of the product rule, which in particular states that

$$(e^{-x}f(x))' = (e^{-x})'f(x) + e^{-x}f'(x) = -e^{-x}f(x) + e^{-x}f'(x).$$

By a slight refinement of Theorem 6.3.2, from (3) we have that

$$\int_{x_0}^{x} (e^{-t}f(t))'dt > 0, \quad \forall x > x_0.$$

On the other hand, from the second part of the fundamental theorem of calculus (Theorem 7.4), we have that

$$\int_{x_0}^{x} (e^{-t}f(t))'dt = e^{-x}f(x) - e^{-x_0}f(x_0) = e^{-x}f(x)$$

since $f(x_0) = 0$. It follows that

$$e^{-x}f(x) > 0, \forall x > x_0.$$

Again because of strict positivity of the exponential function, this is equivalent to the desired conclusion,

$$f(x) > 0, \forall x > x_0.$$

Problem 4 (Series)

(a) Let us start by assuming that the "condensed" series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} < \infty$$

converges. Then

$$\begin{aligned} 0 \leq \sum_{k=1}^{\infty} a_k &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + (a_8 + \ldots + a_{15}) + \ldots \\ &\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + (a_8 + \ldots + a_8) + \ldots \\ &= \sum_{k=0}^{\infty} 2^k a_{2^k}, \end{aligned}$$

and so the series $\sum_{k=1}^{\infty} a_k$ converges as well. Here we just used the fact that the sequence $\{a_k\}$ is nonincreasing. Conversely, assume that the series $\sum_{k=1}^{\infty} a_k$ converges. Then

$$\begin{aligned} 0 \leq \sum_{k=0}^{\infty} 2^k a_{2^k} &= a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots \\ &= a_1 + 2\Big(a_2 + 2a_4 + 4a_8 + \ldots\Big) \\ &\leq a_1 + 2\Big(a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \ldots\Big) \\ &= -a_1 + 2\sum_{k=1}^{\infty} a_k, \end{aligned}$$

and so the series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges as well.

Remarks. Strictly speaking, the various instances of '$\ldots$' make the former proof non-rigorous. One should work with partial sums instead, truncated at N say, and prove each statement by induction on N.

The result we just proved goes in the literature under the name of "Cauchy condensation criterion", and it provides a useful tool to verify the convergence of series which would otherwise be hard to study. As a further application of this criterion, the reader is encouraged to revisit the α-series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}, \quad \text{for } 0 < \alpha < \infty.$$

(b) Yes. The proof outlined in part (a) goes essentially through with only minor modifications (omitted). Under the assumptions of the problem, one can prove more generally that, given a natural number $m \in \mathbb{N} : m \geq 2$, the series $\sum_{k=1}^{\infty} a_k$ converges if and only the series $\sum_{k=0}^{\infty} m^k a_{m^k}$ converges.

(c) Let us use the criterion of part (b) with $m = 10$. For this purpose, define

$$b_n := \frac{1}{nd(n)}$$

and note that

$$d(10^n) = d(1\underbrace{0\ldots0}_{n\ 0's}) = n + 1. \tag{4}$$

It follows that

$$10^n b_{10^n} = \cancel{10^n}\frac{1}{\cancel{10^n}d(10^n)} = \frac{1}{d(10^n)} = \frac{1}{n+1},$$

and so the series

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{nd(n)} \tag{5}$$

converges if and only if the series

$$\sum_{n=0}^{\infty} 10^n b_{10^n} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{k=1}^{\infty} \frac{1}{k}$$

converges. But this is the harmonic series, which is by now well-known to be divergent. It follows that the series (5) diverges as well.

Further define

$$c_n := \frac{1}{nd(n)d(d(n))} \text{ and } d_n := \frac{1}{nd(n)d(d(n))d(d(d(n)))}.$$

Using (4), we have that

$$10^n c_{10^n} = \cancel{10^n}\frac{1}{\cancel{10^n}d(10^n)d(d(10^n)))} = \frac{1}{(n+1)d(n+1)} = b_{n+1}$$

and that

$$10^n d_{10^n} = \cancel{10^n}\frac{1}{\cancel{10^n}d(10^n)d(d(10^n))d(d(d(10^n))))} = \frac{1}{(n+1)d(n+1)d(d(n+1))} = c_{n+1}.$$

From part (b), it follows that the series

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{nd(n)d(d(n))}$$

diverges if and only if the series

$$\sum_{n=0}^{\infty} 10^n c_{10^n} = \sum_{n=0}^{\infty} b_{n+1} = \sum_{n=1}^{\infty} \frac{1}{nd(n)}$$

diverges, which we just verified to be the case. Similarly, the series

$$\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty} \frac{1}{nd(n)d(d(n))d(d(d(n)))}$$

diverges if and only if the series

$$\sum_{n=0}^{\infty} 10^n d_{10^n} = \sum_{n=0}^{\infty} c_{n+1} = \sum_{n=1}^{\infty} \frac{1}{nd(n)d(d(n))}$$

diverges, which we just verified to be the case. The conclusion is that all three series

$$\sum_{n=1}^{\infty} \frac{1}{nd(n)}, \sum_{n=1}^{\infty} \frac{1}{nd(n)d(d(n))}, \text{ and } \sum_{n=1}^{\infty} \frac{1}{nd(n)d(d(n))d(d(d(n)))}$$

diverge, as desired.