Analysis 1, Solutions to problem set 8

Problem 1 (Sequences)

Parts (a) and (c) below will use the fact that *a bounded, monotonically increasing sequence in $\mathbb{X}$ converges to a finite limit.* This is a consequence of Theorem 3.5 together with the fact that an upper bound for a given sequence is greater than or equal to its lim sup.

(a) Start by observing that each element a_n is the sum of positive terms, and therefore the sequence (a_n) is monotonically increasing. In light of Theorem 3.5, it converges. The interesting question is whether it converges to a *finite* limit.

Note that the sequence $(2^{1/k})_{k\geq 1}$ is monotonically decreasing. To see why this is the case, let k be a nonzero natural number. Since $2 \geq 1^k = 1$, we have that $2^{1/k} \geq 1$, and therefore

$$2 \leq 2 \cdot 2^{\frac{1}{k}} = 2^{\frac{k+1}{k}}.$$

This can be rewritten in the form of the desired inequality,

$$2^{\frac{1}{k+1}} \leq 2^{\frac{1}{k}}.$$

It follows that the sequence $(2^{1/k})_{k\geq 1}$ is bounded by its first term, i.e.

$$2^{1/k} \leq 2, \ \forall k \in \mathbb{N} : k \geq 1. \tag{1}$$

As a consequence, we have that, for every natural number $n \geq 1$,

$$a_n = \sum_{k=1}^{n} 2^{1/k} q^k \leq 2 \sum_{k=1}^{n} q^k \leq 2q \sum_{j=0}^{\infty} q^j = \frac{2q}{1-q} < \infty$$

since $0 < q < 1$. Here we are using bound (1) together with our knowledge of the sum of a geometric series.

We conclude that the sequence (a_n) is bounded (by $\frac{2q}{1-q}$). Since it was already known to be monotonically increasing, it converges to a finite limit.

(b) Start by noting that, for $n \geq 1$,

$$a_{n+2} - a_{n+1} = \frac{1}{2}(\cancel{a_n} + a_{n+1}) - \frac{1}{2}(a_{n-1} + \cancel{a_n}) = \frac{1}{2}(a_{n+1} - a_{n-1})$$

and that

$$a_{n+1} - a_{n-1} = \frac{1}{2}(a_{n-1} + a_n) - a_{n-1} = \frac{1}{2}(a_n - a_{n-1}).$$

It follows that

$$a_{n+2} - a_{n+1} = \frac{1}{4}(a_n - a_{n-1}).$$

If n is odd, i.e. $n = 2k+1$ for some natural k, then

$$a_{n+2} - a_{n+1} = \frac{1}{4}(a_n - a_{n-1}) = \frac{1}{4^2}(a_{n-2} - a_{n-3}) = \ldots = \frac{1}{4^{k+1}}(a_1 - a_0) = \frac{1}{4^{k+1}}(1-0) = \frac{1}{2^{n+1}}.$$

Similarly, if $n = 2k$, then

$$a_{n+2} - a_{n+1} = \frac{1}{4}(a_n - a_{n-1}) = \ldots = \frac{1}{4^k}(a_2 - a_1) = \frac{1}{4^k}\left(\frac{1}{2} - 1\right) = -\frac{1}{2^{n+1}}.$$

Both of these statements can be proved by a straightforward induction on k, the details of which are omitted. In any case, we have that

$$|a_{n+2} - a_{n+1}| \leq \frac{1}{2^{n+1}}, \ \forall n \in \mathbb{N}, \tag{2}$$

and this can be used to prove that the sequence (a_n) converges to a finite limit in light of Cauchy's criterion. For that purpose, let $\epsilon > 0$ be arbitrary and choose $n \in \mathbb{N}$ such that

$$\frac{1}{2^n} < \frac{\epsilon}{2}.$$

If $n' > n$ and $m' > n$, then we claim that

$$|a_{n'} - a_{m'}| \leq 2^{-\min\{n',m'\}}. \tag{3}$$

To see why this is true, assume without loss of generality that $n' > m'$, and let $k \in \mathbb{N}$ be such that $n' = m' + k$. By the triangle inequality and estimate (2), we have that

$$\begin{aligned}
|a_{n'} - a_{m'}| &= |a_{m'+k} - a_{m'}| \\
&\leq |a_{m'+k} - a_{m'+k-1}| + |a_{m'+k-1} - a_{m'+k-2}| + \ldots + |a_{m'+1} - a_m'| \\
&\leq \frac{1}{2^{m'+k-1}} + \frac{1}{2^{m'+k-2}} + \ldots + \frac{1}{2^{m'}} \\
&= \frac{1}{2^{m'}} \sum_{j=0}^{k-1} \left(\frac{1}{2}\right)^j \\
&= \frac{1}{2^{m'}} \frac{1 - (1/2)^k}{1 - 1/2} \\
&\leq \frac{1}{2^{m'-1}}.
\end{aligned}$$

This establishes (3). Since $m' > n$, we additionally have that

$$|a_{n'} - a_{m'}| \leq 2^{-(m'-1)} < 2^{-(n-1)} < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude by Theorem 3.9 that the sequence (a_n) converges to a finite limit.

(c) Our first observation is that

$$ba_n < 1, \ \forall n \in \mathbb{N}. \tag{4}$$

This can be proved by induction on n, and this time we include the details. The base case $n = 0$ is equivalent to the assumption $a_0 < 1/b$ since b is a positive real number. Assuming that the statement holds for n, let us prove it for $n + 1$. Compute and estimate:

$$ba_{n+1} = b(2a_n - ba_n^2) = ba_n(2 - ba_n) < 1.$$

Note that the induction hypothesis was used in the last inequality (use the fact that $0 < t < 1 \Rightarrow t(2-t) < 1$ with $t = ba_n$). This establishes (4), and shows that the sequence (a_n) is bounded (by $1/b$).

It follows that

$$a_{n+1} = a_n(2 - ba_n) > a_n \cdot 1 = a_n$$

since $2 - ba_n > 1$ if and only if $ba_n < 1$, which in turn is the content of (4). In particular, the sequence (a_n) is monotonically increasing. In view of Theorem 3.5, it converges. Since we already know it to be bounded, it follows that it converges to a finite limit.

Remark. In this case, one can easily determine the value of the limit $\ell := \lim_{n\to\infty} a_n$. It will have to satisfy the equation $\ell = 2\ell - b\ell^2$, and so $\ell = 0$ or $\ell = 1/b$. Since (a_n) is a monotonically increasing sequence of positive numbers, the first alternative cannot hold, and so $\ell = 1/b$.

Problem 2 (Arithmetic mean)

(a) We will establish the first inequality

$$\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} x_n,$$

and leave the straightforward modifications that deal with the second one as an exercise for the reader.

By letting $c \ge 0$ be a sufficiently large constant and considering the sequence $(x_n + c)$ instead, we lose no generality in assuming that the sequence (x_n) consists of *nonnegative* real numbers. (Caveat: the only situation in which this might pose a problem is if $\limsup_{n\to\infty} x_n = -\infty$. But in this case the sequence $\{x_n\}$ itself must converge to $-\infty$, and it is easy to check that then so does $\{a_n\}$. Thanks to Tobias Terzer for pointing this out.)

Let $M := \limsup_{n\to\infty} x_n$. If $M = \infty$, there is nothing to prove, and so again without loss of generality we may assume that $M < \infty$. In particular, the sequence (x_n) is bounded. Thus there exists $B < \infty$ such that $x_n < B$ for every $n \in \mathbb{N}$. (Carefully note that the estimate $x_n \le M$ fails in general for a given $n \in \mathbb{N}$.)

Now, let $\epsilon > 0$ be given. Choose $n_1 \in \mathbb{N}$ so that

$$x_n < M + \frac{\epsilon}{2} \text{ if } n \ge n_1. \tag{5}$$

(This is possible because $M = \limsup_{n\to\infty} x_n$.) Next, choose $n_2 \in \mathbb{N}$ so that $n_2 > n_1$ and

$$\frac{B}{n} < \frac{\epsilon}{2} \text{ if } n \ge n_2. \tag{6}$$

(This is possible because $\lim_{n\to\infty} \frac{1}{n} = 0$.) Finally, let $N := n_2^2$. If $n > N$, then $n > n_2 > n_1$, and the average defining a_n can be split as follows:

$$a_n = \frac{1}{n}\sum_{k=0}^{n-1} x_k = \frac{1}{n}\Big(\sum_{k=0}^{n_1-1} x_k + \sum_{k=n_1}^{n-1} x_k\Big).$$

The first sum can be estimated as follows:

$$\frac{1}{n}\sum_{k=0}^{n_1-1} x_k < \frac{1}{n}\sum_{k=0}^{n_1-1} B = \frac{Bn_1}{n}.$$

As for the second one, note that (5) implies

$$\frac{1}{n}\sum_{k=n_1}^{n-1} x_k < M + \frac{\epsilon}{2}.$$

All in all, we have that

$$a_n < \frac{Bn_1}{n} + M + \frac{\epsilon}{2}.$$

Since $n > N = n_2^2$ and $n_2 > n_1$, we have that

$$\frac{Bn_1}{n} < \frac{Bn_1}{n_2^2} = \frac{B}{n_2}\frac{n_1}{n_2} < \frac{B}{n_2}\cdot 1 < \frac{\epsilon}{2}$$

because of (6). It follows that

$$a_n < M + \epsilon.$$

For a given $\epsilon > 0$, we have thus produced a threshold $N = N(\epsilon)$ such that, for every $n > N$,

$$a_n < M + \epsilon.$$

It follows that

$$\limsup_{n\to\infty} a_n = \inf_{n\in\mathbb{N}}\Big(\sup_{m\ge n} a_m\Big) \le \sup_{m\ge N+1} a_m = \sup_{m>N} a_m \le M + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that

$$\limsup_{n\to\infty} a_n \le M,$$

as desired.

(b) Let $x \in \mathbb{R}$, and assume that the sequence (x_n) converges to x. This means that

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x.$$

From part (a) it follows that

$$x = \liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} x_n = x.$$

In particular, all the inequalities in the last chain of inequalities are equalities, and

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = x.$$

This implies the desired conclusion.

In general, this implication cannot be reversed. We provide an example. Consider the sequence (x_n) defined via

$$x_n = \begin{cases} 1 & \text{if } n = 2^k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\liminf_{n\to\infty} x_n = 0 < 1 = \limsup_{n\to\infty} x_n$, and so the sequence (x_n) does not converge. We will be done once we show that the sequence of means (a_n) does converge. We will show more, namely, that $\lim_{n\to\infty} a_n = 0$.

With this purpose in mind, recall the function L defined in Problem 3, Übungsblatt 6 via

$$L(n) := \max\{j \in \mathbb{N} : 2^j \leq n\}.$$

Calculate and estimate:

$$a_n = \frac{1}{n}\sum_{k=0}^{n-1} x_k = \frac{1}{n}\Bigg(\sum_{\substack{0\leq k<n: \\ \exists j\in\mathbb{N}: k=2^j}} 1 \Bigg) = \frac{\#\{k \in \mathbb{N} : k < n \text{ and } k = 2^j \text{ for some } j \in \mathbb{N}\}}{n}$$

$$= \frac{\#\{j \in \mathbb{N} : 2^j < n\}}{n} \leq \frac{L(n)}{n}.$$

From Problem 3(a) of Übungsblatt 6 we know that

$$2^{L(n)} \leq n < 2^{L(n)+1},$$

and this can be used to check that the sequence $(\frac{L(n)}{n})$ decreases monotonically to 0. Since

$$0 \leq a_n \leq \frac{L(n)}{n},$$

it follows that $\lim_{n\to\infty} a_n = 0$ as well.

Problem 3 (Cauchy products)

(a) In light of Problem 3(c) from Übungsblatt 6, we already know that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$$

diverges. It follows that the sequence a is not absolutely summable. To show that it is a summable sequence, we make use of Leibniz's criterion (Theorem 4.14). For that purpose, define $g(n) := \frac{1}{\sqrt{n}}$ whenever $n \in \mathbb{N} \setminus \{0\}$. We have to check that (i) $g(n+1) \leq g(n)$ for every $n \geq 1$, and that

$$(ii) \quad \liminf_{n\to\infty} g(n) = 0.$$

Property (i) is immediate since

$$g(n+1) \leq g(n) \Leftrightarrow \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} \Leftrightarrow \sqrt{n} \leq \sqrt{n+1} \Leftrightarrow n \leq n+1.$$

Moreover, since the sequence $(\sqrt{n})$ increases monotonically and unboundedly, it follows that the sequence $(\frac{1}{\sqrt{n}})$ converges monotonically to 0. In particular, property (ii) holds. This establishes the summability of the sequence a, i.e. the convergence of the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} < \infty.$$

(b) Start by noting that

$$c_n = \sum_{k=1}^{n-1} a_k a_{n-k} = \sum_{k=1}^{n-1} \frac{(-1)^k}{\sqrt{k}} \frac{(-1)^{n-k}}{\sqrt{n-k}} = (-1)^n \sum_{k=1}^{n-1} \frac{1}{\sqrt{k(n-k)}}. \tag{7}$$

If n is even (and nonzero), then

$$\min_{1\le k\le n-1} \frac{1}{\sqrt{k(n-k)}} = \frac{2}{n}. \tag{8}$$

To see why this is the case, note that the AM-GM inequality implies that

$$k(n-k) \le \frac{k^2+(n-k)^2}{2}$$

for every natural $k \in \{1,2,\ldots,n-1\}$. If n is even and $k = n/2$, then this inequality becomes an equality. (Moreover, this choice of k is actually the only one for which the inequality becomes an equality.) It follows that

$$\max_{1\le k\le n-1} k(n-k) = \frac{n}{2}\Big(n - \frac{n}{2}\Big) = \Big(\frac{n}{2}\Big)^2,$$

and so

$$\max_{1\le k\le n-1} \sqrt{k(n-k)} = \frac{n}{2}.$$

This implies the claimed identity (8).

Going back to (7), we see that, if $n > 0$ is even, then $(-1)^n = 1$ and

$$c_n = \sum_{k=1}^{n-1} \frac{1}{\sqrt{k(n-k)}} \ge \frac{2}{n} \sum_{k=1}^{n-1} 1 = 2\Big(1 - \frac{1}{n}\Big).$$

Thus $\limsup_{n\to\infty} c_n \ge 2$.

If the series $\sum_{n=1}^{\infty} c_n$ were to converge to a finite limit, then we would necessarily have that $\lim_{n\to\infty} c_n = 0$. (The proof of this last statement is left as an important exercise for the reader.) Since $\limsup_{n\to\infty} c_n = 2 \neq 0$, we conclude that the series $\sum_{n=1}^{\infty} c_n$ diverges.

Problem 4 (Another – less surprising? – convergent series)

We provide a solution which assumes the knowledge of the *fundamental theorem of arithmetic*. This theorem states that any natural number greater than 1 is either a prime number itself or the product of prime numbers. Although the order of the primes in the second case is arbitrary, the primes themselves are not. For example,

$$1200 = 2^4 \cdot 3^1 \cdot 5^2 = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 = 5 \cdot 2 \cdot 3 \cdot 2 \cdot 5 \cdot 2 \cdot 2 = \ldots$$

The theorem is stating two things: first, that 1200 can be represented as a product of primes, and second, that no matter how this is done, there will always be four 2s, one 3, two 5s, and no other primes in the product.

From the fundamental theorem of arithmetic, it follows that any natural number $n \in \mathcal{M}$ (i.e. which is *not* divisible by any prime factor different from 2 or 5) must necessarily be of the form $n = 2^k \cdot 5^\ell$ for *unique* natural numbers k and ℓ. Let $m = k + \ell$ be the number of prime factors (counted with multiplicity) appearing in a given $n \in \mathcal{M}$, and let $\mathcal{M}_m$ be the subset of $\mathcal{M}$ consisting of all numbers n of the form $n = 2^k \cdot 5^\ell$ for some natural numbers k, ℓ satisfying $k + \ell = m$. In particular,

$$\sum_{n\in\mathcal{M}} \frac{1}{n} = \sum_{m=0}^{\infty} \Big(\sum_{n\in\mathcal{M}_m} \frac{1}{n} \Big), \tag{9}$$

where the inner sum can be written as

$$\sum_{n\in\mathcal{M}_m}\frac{1}{n}=\sum_{\substack{k,\ell\in\mathbb{N}:\\k+\ell=m}}\frac{1}{2^k\cdot 5^\ell}=\sum_{k=0}^{m}\frac{1}{2^k}\frac{1}{5^{m-k}}. \tag{10}$$

A nice observation consists of recognizing this as the general term of a simple Cauchy product, as defined in Problem 3 above. With this goal in mind, consider the geometric series $\sum_{m\in\mathbb{N}}\frac{1}{2^m}$ and $\sum_{m\in\mathbb{N}}\frac{1}{5^m}$. By definition, their Cauchy product equals the series $\sum_{m=0}^{\infty}c_m$ whose general term is given by

$$c_m=\sum_{k=0}^{m}\frac{1}{2^k}\frac{1}{5^{m-k}}.$$

In view of (9) and (10), it follows that

$$\sum_{n\in\mathcal{M}}\frac{1}{n}=\sum_{m=0}^{\infty}\Big(\sum_{n\in\mathcal{M}_m}\frac{1}{n}\Big)=\sum_{m=0}^{\infty}c_m=\Big(\sum_{m=0}^{\infty}\frac{1}{2^m}\Big)\cdot\Big(\sum_{m=0}^{\infty}\frac{1}{5^m}\Big)=\frac{1}{1-1/2}\cdot\frac{1}{1-1/5}=2\cdot\frac{5}{4}=\frac{5}{2}.$$

In particular, the series $\sum_{n\in\mathcal{M}}\frac{1}{n}$ converges and its value equals $\frac{5}{2}$, as desired.

Remark on the proof of the fundamental theorem of arithmetic. There are two parts to the proof, existence and uniqueness of the promised factorization of a given natural number n into primes. Existence can be proved via *complete* induction on n. Uniqueness is typically established resorting to some version of Euclid's algorithm, even if elementary proofs that do not require it are available at the expense of greater length.