Analysis 1, Solutions to problem set 7

Problem 1 (Sequences: $\sup, \inf, \limsup, \liminf$)

In what follows, let n be a natural number greater than or equal to 1.
A preliminary observation, common to all the items below, is that $(-1)^n = 1$ if n is even and $(-1)^n = -1$ if n is odd.

(a) In light of the preliminary observation, note that

$$x_n = \begin{cases} (2+\frac{3}{n}) & \text{if } n \text{ is odd,} \\ -(2+\frac{3}{n}) & \text{if } n \text{ is even.} \end{cases}$$

The sequence $(2+\frac{3}{n})_{n\geq 1}$ is monotonically decreasing, and so the sequence $(-(2+\frac{3}{n}))_{n\geq 1}$ is monotonically increasing. Moreover we have that

$$\lim_{n\to\infty}\left(2+\frac{3}{n}\right) = \inf_{n\geq 1}\left(2+\frac{3}{n}\right) = 2 \text{ and } \lim_{n\to\infty}\left\{-\left(2+\frac{3}{n}\right)\right\} = \sup_{n\geq 1}\left\{-\left(2+\frac{3}{n}\right)\right\} = -2.$$

The proof of these facts entirely parallels the proof that $\lim_{n\to\infty}\frac{1}{n} = 0$, and as such will be omitted.
It follows that

$$\sup_{n\geq 1} x_n = x_1 = 2 + \frac{3}{1} = 5$$

and that

$$\inf_{n\geq 1} x_n = x_2 = -\left(2+\frac{3}{2}\right) = -\frac{7}{2}.$$

Moreover,

$$\sup_{n\geq m} x_n = \begin{cases} 2+\frac{3}{m} & \text{if } m \text{ is odd,} \\ 2+\frac{3}{m+1} & \text{if } m \text{ is even,} \end{cases}$$

and

$$\inf_{n\geq m} x_n = \begin{cases} -(2+\frac{3}{m+1}) & \text{if } m \text{ is odd,} \\ -(2+\frac{3}{m}) & \text{if } m \text{ is even.} \end{cases}$$

It follows that

$$\limsup_{n\to\infty} x_n = \inf_{m\geq 1}\left(\sup_{n\geq m} x_n\right) = \left\{\begin{array}{ll} \inf_{m\geq 1}\left(2+\frac{3}{m}\right) & \text{if } m \text{ is odd} \\ \inf_{m\geq 1}\left(2+\frac{3}{m+1}\right) & \text{if } m \text{ is even} \end{array}\right\} = 2.$$

and that

$$\liminf_{n\to\infty} x_n = \sup_{m\geq 1}\left(\inf_{n\geq m} x_n\right) = \left\{\begin{array}{ll} \sup_{m\geq 1}\left\{-\left(2+\frac{3}{m+1}\right)\right\} & \text{if } m \text{ is odd} \\ \sup_{m\geq 1}\left\{-\left(2+\frac{3}{m}\right)\right\} & \text{if } m \text{ is even} \end{array}\right\} = -2.$$

(b) The crucial observation is that

$$x_n = \begin{cases} 6 & \text{if } n = 4k+1 \text{ for some } k\in\mathbb{N}, \\ -4 & \text{if } n = 4k+2 \text{ for some } k\in\mathbb{N}, \\ 0 & \text{if } n = 4k+3 \text{ for some } k\in\mathbb{N}, \\ 2 & \text{if } n = 4k+4 \text{ for some } k\in\mathbb{N}. \end{cases}$$

Assuming this, it follows at once that

$$\sup_{n\geq 1} x_n = \limsup_{n\to\infty} x_n = 6$$

and that

$$\inf_{n\geq 1} x_n = \liminf_{n\to\infty} x_n = -4.$$

The proof of the claim amounts to a straightforward analysis of the four different cases that arise. For instance, if $n = 4k+1$ for some natural number k, then

$$\begin{aligned} x_n &= 1 + 2(-1)^{n+1} + 3(-1)^{\frac{n(n-1)}{2}} = 1 + 2(-1)^{(4k+1)+1} + 3(-1)^{\frac{(4k+1)(4k)}{2}} \\ &= 1 + 2(-1)^{2(2k+1)} + 3(-1)^{2k(4k+1)} = 1 + 2\cdot 1 + 3\cdot 1 = 6 \end{aligned}$$

since both $2(2k+1)$ and $2k(4k+1)$ are even numbers. The analysis of the other three cases is similar and is left to the reader.

(c) Note that

$$x_n = \begin{cases} n & \text{if } n \text{ is even,} \\ \frac{1}{n} & \text{if } n \text{ is odd.} \end{cases}$$

In particular, the subsequence $(x_{2n} = 2n)$ is monotonically increasing and unbounded, and

$$\sup_{n\geq 1} x_n = \limsup_{n\to\infty} x_n = \infty.$$

On the other hand, the subsequence $(x_{2n+1} = \frac{1}{2n+1})$ is monotonically decreasing and satisfies

$$\lim_{n\to\infty} x_{2n+1} = 0.$$

It follows that

$$\inf_{n\geq 1} x_n = \liminf_{n\to\infty} x_n = 0.$$

Problem 2 (Series)

(a) Start by noticing that, for every natural number $n \geq 1$,

$$\frac{(n+1)^n}{n^{n+1}} = \frac{1}{n}\frac{(n+1)^n}{n^n} = \frac{1}{n}\Big(1+\frac{1}{n}\Big)^n.$$

In Problem 4(a) from Übungsblatt 6 we showed that the sequence $\alpha_n := (1+\frac{1}{n})^n$ is monotonically increasing. In particular, $\alpha_n \geq \alpha_1$ for every $n \geq 1$. Since $\alpha_1 = 2$, it follows that

$$\frac{(n+1)^n}{n^{n+1}} = \frac{1}{n}\Big(1+\frac{1}{n}\Big)^n \geq \frac{2}{n},$$

and so

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}} \geq \sum_{n=1}^{\infty} \frac{2}{n} = 2\sum_{n=1}^{\infty}\frac{1}{n}. \tag{1}$$

We know from Problem 3(c) of Übungsblatt 6 that the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. From (1), it follows that the series

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}} = \infty$$

diverges as well.

(b) For $n \in \mathbb{N}$, define $a_n := \frac{n^3}{n!}$. Let us compute the quotient

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{(n+1)!}\frac{n!}{n^3} = \frac{\cancel{(n+1)}(n+1)^2}{\cancel{(n+1)}n!}\frac{n!}{n^3} = \frac{(n+1)^2}{\cancel{n!}}\frac{\cancel{n!}}{n^3} = \frac{(n+1)^2}{n^3} = \frac{1}{n}\Big(1+\frac{1}{n}\Big)^2.$$

Observe that the sequence

$$\frac{a_{n+1}}{a_n} = \frac{1}{n}\Big(1+\frac{1}{n}\Big)^2 = \frac{1}{n}\Big(1+\frac{2}{n}+\frac{1}{n^2}\Big) = \frac{1}{n}+\frac{2}{n^2}+\frac{1}{n^3}$$

is monotonically decreasing and that

$$\frac{a_4}{a_3} = \frac{1}{3}\Big(1+\frac{1}{3}\Big)^2 = \frac{16}{27} < 1.$$

It follows that

$$\frac{a_{n+1}}{a_n} \leq \frac{16}{27}$$

for every $n \geq 3$, and so the conditions of Problem 2(b), Übungsblatt 6 are fulfilled (for $N = 3$ and $y = 16/27$). We can use that result to establish the convergence of the sequence

$$\sum_{n=0}^{\infty} \frac{n^3}{n!} < \infty.$$

(c) We proceed similarly to part (b). For $n \in \mathbb{N}$, define $b_n := \frac{(n!)^2}{(2n)!}$. Then

$$\frac{b_{n+1}}{b_n} = \frac{((n+1)!)^2}{(2(n+1))!}\frac{(2n)!}{(n!)^2} = \frac{(n+1)\cancel{n!}(n+1)\cancel{n!}}{(2n+2)!}\frac{(2n)!}{\cancel{n!}\cancel{n!}} = (n+1)^2\frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} = \frac{(n+1)^2}{(2n+2)(2n+1)}.$$

Again, the sequence

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{2}\frac{n+1}{2n+1} = \frac{1}{2}\Big(1 - \frac{n}{2n+1}\Big) = \frac{1}{2}\Big(1 - \frac{1}{2+\frac{1}{n}}\Big)$$

is monotonically decreasing, and

$$\frac{b_1}{b_0} = \frac{(0+1)^2}{(2\cdot 0+2)(2\cdot 0+1)} = \frac{1}{2} < 1.$$

Problem 2(b) from Übungsblatt 6 (with $N = 0$ and $y = 1/2$) implies that the series

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} < \infty$$

converges.

(d) Start by observing that, for every natural number $n \geq 1$,

$$\frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{n(\sqrt{n+1}+\sqrt{n})} = \frac{(\cancel{n}+1)-\cancel{n}}{n(\sqrt{n+1}+\sqrt{n})} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})}.$$

Moreover, we have that

$$\frac{1}{n(\sqrt{n+1}+\sqrt{n})} \leq \frac{1}{n\sqrt{n}}$$

since $\sqrt{n} \leq \sqrt{n+1}+\sqrt{n}$. Since this holds for every natural $n \geq 1$, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \leq \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}.$$

From Problem 3(c), Übungsblatt 6, we know that the series

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty$$

converges since $3/2 > 1$. It follows that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(\sqrt{n+1}+\sqrt{n})} < \infty$$

converges as well.

Problem 3 (Fubini without Tonelli)

Let $f = (f_n(m))_{m,n}$, where $m \in \mathbb{N}$ designates the number of the (horizontal) row and $n \in \mathbb{N}$ designates the number of the (vertical) column, be the following infinite matrix:

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \cdots \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ -\frac{1}{4} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In other words,

$$f_n(m) = \begin{cases} 2^{m-n} & \text{if } n > m, \\ 0 & \text{if } n = m, \\ -2^{n-m} & \text{if } n < m. \end{cases}$$

It follows that, for $m = 0, 1, 2, \ldots$,

$$\begin{aligned} \sum_{n=0}^{\infty} f_n(m) &= \cancel{\sum_{n=0}^{m-1}(-2^{n-m})} + 0 + \cancel{\sum_{n=m+1}^{2m} 2^{m-n}} + \sum_{n=2m+1}^{\infty} 2^{m-n} = \sum_{n=2m+1}^{\infty} 2^{m-n} \\ &= \sum_{k=1}^{\infty} 2^{-m-k} = 2^{-m}\sum_{k=1}^{\infty} 2^{-k} = 2^{-m}\frac{1}{2}\sum_{k=0}^{\infty} 2^{-k} = 2^{-m}\frac{1}{2}\frac{1}{1-1/2} = 2^{-m}, \end{aligned}$$

where in the passage from the first to the second line we changed summation index $k = n - 2m$. It follows that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f_n(m) = \sum_{m=0}^{\infty} 2^{-m} = \frac{1}{1-1/2} = 2.$$

On the other hand, for $n = 0, 1, 2, \ldots$,

$$\sum_{m=0}^{\infty} f_n(m) = -2^{-n}\frac{1}{2}\sum_{k=0}^{\infty} 2^{-k} = -2^{-n},$$

and so

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_n(m) = \sum_{n=0}^{\infty}(-2^{-n}) = -2.$$

Since $2 \neq -2$, the sequence (of sequences) f does the required job. Note that, in light of Theorem 3.40 (Fubini-Tonelli), such a result could never hold if instead we considered sequences of (extended) positive real numbers.

Problem 4 (A – perhaps surprising? – convergent series)

For a natural number $n \geq 1$, define the set

$$S_n := \{k \in \mathbb{N} : k \geq 1 \text{ and } k \text{ has exactly } n \text{ digits and none of them is } 9\}.$$

Our first claim is that *the set S_n has $8 \cdot 9^{n-1}$ elements.* This can be verified via induction on n. For $n = 1$, we have that

$$S_1 = \{1, 2, 3, 4, 5, 6, 7, 8\},$$

and so S_1 has $8 = 8 \cdot 9^{1-1}$ elements, as desired. Let us assume that the set S_n has $8 \cdot 9^{n-1}$ elements, and prove that the set S_{n+1} has $8 \cdot 9^n$ elements. Any elements of S_{n+1} can be obtained in a unique way from an element of S_n by concatenating a single digit to its end. Conversely, any element of S_n gives rise to an element of S_{n+1} by concatenation of a single digit to its end. For each element of S_n, there are 9 possibilities for doing this (one chooses one of 0,1,2,3,4,5,6,7,8). It follows that the number of elements of S_{n+1} is exactly 9 times the number of elements of S_n, which by induction hypothesis is $8 \cdot 9^{n-1}$. Therefore the number of elements of the set S_{n+1} is $9 \cdot (8 \cdot 9^{n-1}) = 8 \cdot 9^n$, as desired.
Our next observation is that *any element of S_n is at least 10^{n-1}*. This follows from noting that

$$10^{n-1} = \underbrace{100\ldots0}_{n \text{ digits}} \in S_n,$$

and that

$$10^{n-1} - 1 = \underbrace{99\ldots9}_{n-1 \text{ digits}} \in S_{n-1}.$$

As a consequence, the elements of the set S_n contribute to the original sum by less that $\frac{8 \cdot 9^{n-1}}{10^{n-1}}$. Indeed, denoting the number of elements of the set S_n by $\#S_n$, we have that

$$\sum_{k\in S_n} \frac{\epsilon_n}{n} \leq \sum_{k\in S_n} \frac{1}{10^{n-1}} = \frac{1}{10^{n-1}}\sum_{k\in S_n} 1 = \frac{1}{10^{n-1}}(\#S_n) = \frac{8 \cdot 9^{n-1}}{10^{n-1}}.$$

We can finally conclude that

$$\sum_{n=1}^{\infty} \frac{\epsilon_n}{n} = \sum_{n=1}^{\infty} \Big(\sum_{k \in S_n} \frac{\epsilon_n}{n} \Big) \leq \sum_{n=1}^{\infty} \frac{8 \cdot 9^{n-1}}{10^{n-1}} = 8 \sum_{n=1}^{\infty} \Big(\frac{9}{10}\Big)^{n-1} = 8 \sum_{m=1}^{\infty} \Big(\frac{9}{10}\Big)^{m} = 8 \frac{1}{1 - 9/10} = 80 < \infty.$$

Note that we proved a stronger result. Not only does the series converge, but we also have the quantitative estimate

$$\sum_{n=1}^{\infty} \frac{\epsilon_n}{n} \leq 80.$$

As a final remark, let us mention that this series is an example of what in the literature goes under the name of "Kempner series". It can be shown that, up to 10 decimals, the actual sum is 22.9206766193, but the methods for proving that result are beyond the scope of this class.