Analysis 1, Solutions to problem set 6

Problem 1 (Cauchy criterion for converging sequences)

Let us start by proving that a sequence that converges to a finite limit satisfies Cauchy's criterion. With that purpose in mind, let $f : \mathbb{N} \to \mathbb{X}$ be a sequence for which

$$\liminf_{n\to\infty} f(n) = \limsup_{n\to\infty} f(n) < \infty. \tag{1}$$

As usual, consider the auxiliary sequences $g, h : \mathbb{N} \to \mathbb{X}$ defined by

$$\begin{aligned} g(n) &:= \inf\{f(m) : m \geq n\} \\ h(n) &:= \sup\{f(m) : m \geq n\}. \end{aligned}$$

The sequence g is monotonically increasing and the sequence h is monotonically decreasing. Let

$$A := \sup_{n\in\mathbb{N}} g(n) \text{ and } B := \inf_{n\in\mathbb{N}} h(n).$$

By hypothesis, $A = B < \infty$. By Theorem 2.36 and a straightforward modification thereof (for monotonically decreasing sequences), we know that

$$\forall \epsilon > 0, \exists n_1 \in \mathbb{N} : \forall m > n_1, \;\; g(m) \leq A < g(m) + \epsilon, \tag{2}$$

$$\forall \epsilon > 0, \exists n_2 \in \mathbb{N} : \forall m' > n_2, \;\; h(m') < A + \epsilon \leq h(m') + \epsilon. \tag{3}$$

Moreover we have that, for every $n \in \mathbb{N}$,

$$g(n) \leq f(n) \leq h(n).$$

Now, let $\epsilon > 0$ be given. Since $\epsilon/2 > 0$ as well, we know from (2) and (3) that there exist natural numbers $n_1, n_2 \in \mathbb{N}$ with the following properties:

$$\begin{aligned} g(m) \leq A < g(m) + \epsilon/2 \text{ if } m > n_1, \text{ and} \\ h(m') < A + \epsilon/2 \leq h(m') + \epsilon/2 \text{ if } m' > n_2. \end{aligned}$$

Let $n := \max\{n_1, n_2\}$. Then, for $m, m' > n$, we have that

$$\begin{aligned} A < g(m) + \epsilon/2 \leq f(m) + \epsilon/2 \leq h(m) + \epsilon/2 < A + \epsilon, \\ A < g(m') + \epsilon/2 \leq f(m') + \epsilon/2 \leq h(m') + \epsilon/2 < A + \epsilon, \end{aligned}$$

and so in particular

$$\begin{aligned} &f(m) \leq A + \epsilon/2, \text{ and} \\ &A < f(m') + \epsilon/2. \end{aligned}$$

Since $A < \infty$, we conclude from adding these two inequalities that

$$f(m) < f(m') + \epsilon.$$

The proof that $f(m') < f(m) + \epsilon$ is analogous and left to the reader.

Now we prove that a sequence that satisfies Cauchy's criterion converges to a finite limit. With that purpose in mind, let $f : \mathbb{N} \to \mathbb{X}$ be a sequence satisfying

$$\forall \epsilon > 0 \exists n \in \mathbb{N} \forall m \geq n, m' \geq n : f(m) < f(m') + \epsilon \land f(m') < f(m) + \epsilon, \tag{4}$$

and consider the auxiliary sequences g and h as defined above. Let $\epsilon > 0$ be given. Applying (4), we conclude the existence of a natural number $n_3 \in \mathbb{N}$ such that

$$f(m) < f(m') + \epsilon/4 \wedge f(m') < f(m) + \epsilon/4 \text{ if } m, m' > n_3.$$

In what follows, we will summarize this by writing more succinctly

$$|f(m) - f(m')| < \epsilon/4 \text{ if } m, m' > n_3. \tag{5}$$

We claim that $|g(n_3) - h(n_3)| < \epsilon/2$. This implies that $|g(n) - h(n)| < \epsilon/2$ for every $n \geq n_3$. In fact, since g is monotonically increasing and h is monotonically decreasing, if $n \geq n_3$, then

$$g(n_3) \leq g(n) \leq f(n) \leq h(n) \leq h(n_3),$$

and so

$$|h(n) - g(n)| \leq |h(n_3) - g(n_3)| < \epsilon/2.$$

It remains to verify the claim. Suppose not, i.e. suppose that $|h(n_3) - g(n_3)| \geq \epsilon/2$. Since $h(n_3) \geq g(n_3)$, this can be rewritten as

$$h(n_3) \geq g(n_3) + \epsilon/2.$$

By definition, $g(n_3) = \inf\{f(m) : m \geq n_3\}$ and $h(n_3) = \sup\{f(m) : m \geq n_3\}$. Recalling that the infimum of a set is its largest lower bound, and the supremum of a set is its smallest upper bound, we see that, for every $\delta > 0$, and so in particular for $\delta := \epsilon/100$, there exist natural numbers $m, m' > n_3$ such that

$$g(n_3) + \delta > f(m) \text{ and } f(m') + \delta > h(n_3).$$

It follows that

$$f(m) + \epsilon/2 < g(n_3) + \delta + \epsilon/2 \leq h(n_3) + \delta < f(m') + 2\delta.$$

Recalling that $\delta = \epsilon/100$, it follows that

$$f(m') + \epsilon/50 = f(m') + 2\delta > f(m) + \epsilon/2,$$

and so

$$f(m') > f(m) + 12\epsilon/25 > f(m) + \epsilon/4,$$

which is absurd in light of (5). This establishes the claim.
So now we know that

$$|g(n) - h(n)| < \epsilon/2 \text{ for every } n \geq n_3. \tag{6}$$

Denoting as before $A := \sup_n g(n)$ and $B := \inf_n h(n)$, we will be done once we show that $A = B$. It will suffice to show that $A \geq B$, for the reverse inequality holds for a general sequence f in light of Theorem 3.4. We claim that $A + \epsilon \geq B$, and establish this by contradiction. Suppose not, i.e. suppose

$$A + \epsilon < B.$$

Since $g(n_3) \leq A$ and $B \leq h(n_3)$, we would then have that

$$g(n_3) + \epsilon \leq A + \epsilon < B < h(n_3),$$

and so

$$g(n_3) + \epsilon < h(n_3),$$

in contradiction to (6). The contradiction shows that $A + \epsilon \geq B$. Since $\epsilon > 0$ could have been chosen arbitrarily small, this implies that $A \geq B$, and so $A = B$, as desired. The proof is now complete.

Problem 2 (Series, quotients and inequalities)

(a) From Problem 2(c) in Übungsblatt 2 we have an expression for the partial sums of the series we are trying to compute, namely

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}.$$

Note carefully that up to now only the condition $x \neq 1$ was used. Now, since $0 < x < 1$, we have that $\lim_{n\to\infty} x^{n+1} = 0$, and so

$$\sum_{n=0}^{\infty} x^n = \lim_{n\to\infty} \sum_{k=0}^{n} x^k = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x}.$$

(b) Let $N \in \mathbb{N}$ be as promised by the hypothesis of the problem. We claim that $a_{N+k} \leq y^k a_N$ for every $k \in \mathbb{N}$. This can be verified by induction, the base case $k = 0$ amounting to the identity $a_N = a_N$. Assuming the statement has been proved for k, compute:

$$a_{N+(k+1)} = a_{(N+k)+1} \leq y a_{N+k} \leq y(y^k a_N) = y^{k+1} a_N,$$

as desired. Here we used the assumption of the problem and the induction hypothesis, respectively.

It follows that:

$$\begin{aligned}\sum_{n=0}^{\infty} a_n &= \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{k=0}^{\infty} a_{N+k} \\ &\leq \sum_{n=0}^{N-1} a_n + a_N \sum_{k=0}^{\infty} y^k = \sum_{n=0}^{N-1} a_n + \frac{a_N}{1-y}. \end{aligned} \tag{7}$$

Here, we just changed indices $n = N + k$, applied the claim, and appealed to part (a) (which is legitimate since $0 < y < 1$). The right-hand side of (7) is clearly a finite number, and the proof is complete.

(c) By the elementary inequality proved in Problem 3(d) from Übungsblatt 1 we have that, for each $n \in \mathbb{N}$,

$$a_n b_n \leq \frac{1}{2} a_n^2 + \frac{1}{2} b_n^2.$$

It follows that

$$\sum_{n=0}^{\infty} a_n b_n \leq \frac{1}{2} \sum_{n=0}^{\infty} a_n^2 + \frac{1}{2} \sum_{n=0}^{\infty} b_n^2, \tag{8}$$

and so the series on the left-hand side of this inequality is finite provided both of the series on the right-hand side are finite, as desired.

For the second part, let us appeal to a common "normalization" trick. If neither of the sequences a, b is made up of all zeros (in which case the inequality we are after holds trivially), then we can introduce the modified sequences

$$\widetilde{a}_n = a_n / \Big(\sum_{k=0}^{\infty} a_k^2 \Big)^{1/2} \text{ and } \widetilde{b}_n = b_n / \Big(\sum_{k=0}^{\infty} b_k^2 \Big)^{1/2}.$$

These are "normalized" in the sense that

$$\sum_{n=0}^{\infty} \widetilde{a}_n^2 = \sum_{n=0}^{\infty} \Big\{ a_n^2 / \Big(\sum_{k=0}^{\infty} a_k^2 \Big) \Big\} = \frac{\sum_{n=0}^{\infty} a_n^2}{\sum_{k=0}^{\infty} a_k^2} = 1,$$

and similarly

$$\sum_{n=0}^{\infty} \widetilde{b}_n^2 = 1.$$

Applying (8) to the sequences $\widetilde{a}$ and $\widetilde{b}$ yields

$$\sum_{n=0}^{\infty} \widetilde{a}_n \widetilde{b}_n \leq \frac{1}{2} \sum_{n=0}^{\infty} \widetilde{a}_n^2 + \frac{1}{2} \sum_{n=0}^{\infty} \widetilde{b}_n^2,$$

which can be rewritten in terms of the original sequences a and b as

$$\sum_{n=0}^{\infty}\Big\{a_n/\Big(\sum_{k=0}^{\infty}a_k^2\Big)^{1/2}\Big\}\Big\{b_n/\Big(\sum_{k=0}^{\infty}b_k^2\Big)^{1/2}\Big\}\leq\frac{1}{2}\cdot 1+\frac{1}{2}\cdot 1=1.$$

Clearing denominators, we find our old friend Cauchy-Schwarz, this time covering the case of possibly infinite sequences:

$$\sum_{n=0}^{\infty}a_nb_n\leq\Big(\sum_{n=0}^{\infty}a_n^2\Big)^{1/2}\Big(\sum_{n=0}^{\infty}b_n^2\Big)^{1/2}.$$

(d) Using the Cauchy-Schwarz inequality proved in part (c), we have that

$$\sum_{n=1}^{\infty}\frac{a_n}{n}\leq\Big(\sum_{n=1}^{\infty}a_n^2\Big)^{1/2}\Big(\sum_{n=1}^{\infty}\frac{1}{n^2}\Big)^{1/2}.$$

Since the series $\sum_{n=0}^{\infty}a_n^2<\infty$ by hypothesis, we have that $\sum_{n=1}^{\infty}a_n^2<\infty$, and so will be done once we check that $\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty$. For that purpose, let us follows the hint. Start by noting that, if $n\geq 2$,

$$\frac{1}{n^2}\leq\frac{1}{n(n-1)}.$$

On the other hand, for the same range of n,

$$\frac{1}{n(n-1)}=\frac{1}{n-1}-\frac{1}{n}.$$

It follows that the series with general term $(n(n-1))^{-1}$ telescopes, i.e. its partial sums obey

$$\begin{aligned}\sum_{n=2}^{N}\frac{1}{n(n-1)}=\sum_{n=2}^{N}\Big(\frac{1}{n-1}-\frac{1}{n}\Big)&=\Big(1-\frac{1}{2}\Big)+\Big(\frac{1}{2}-\frac{1}{3}\Big)+\Big(\frac{1}{3}-\frac{1}{4}\Big)+\ldots+\frac{1}{N-2}-\frac{1}{N-1}+\Big(\frac{1}{N-1}-\frac{1}{N}\Big)\\&=1+\underbrace{\Big(\frac{1}{2}-\frac{1}{2}\Big)}_{=0}+\underbrace{\Big(\frac{1}{3}-\frac{1}{3}\Big)}_{=0}+\ldots+\underbrace{\Big(\frac{1}{N-1}-\frac{1}{N-1}\Big)}_{=0}-\frac{1}{N}\\&=1-\frac{1}{N}.\end{aligned}$$

It follows that

$$\sum_{n=2}^{\infty}\frac{1}{n(n-1)}=\lim_{N\to\infty}\sum_{n=2}^{N}\frac{1}{n(n-1)}=\lim_{N\to\infty}\Big(1-\frac{1}{N}\Big)=1,$$

and so

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\sum_{n=2}^{\infty}\frac{1}{n^2}\leq 1+\sum_{n=2}^{\infty}\frac{1}{n(n-1)}=1+1=2<\infty.$$

Problem 3 (α–test)

(a) Let n be a natural number greater than or equal to 1. By definition of maximum, we have that $L(n) \in \{k \in \mathbb{N} : 2^k \le n\}$, and so

$$2^{L(n)} \le n.$$

On the other hand,

$$n < 2^{L(n)+1},$$

for if $2^{L(n)+1} \le n$, then $L(n)+1$ would be an element of the set $\{k \in \mathbb{N} : 2^k \le n\}$ which is strictly larger than $L(n)$, thereby contradicting the assumption that $L(n)$ is the maximum of that set. Therefore we have that

$$2^{L(n)} \le n < 2^{L(n)+1},$$

as desired.

(b) The crucial combinatorial observation is that, for $j \in \{0, 1, \ldots, k-1\}$, there are 2^j numbers for which $L(n) = j$. In fact, these are the elements of the set $\{2^j, 2^j+1, 2^j+2, \ldots, 2^j+(2^j-1)\}$. Additionally note that

$$\sum_{j=0}^{k-1} 2^j = \frac{1-2^k}{1-2} = 2^k - 1,$$

and so the original sum can be decomposed as

$$\sum_{n=1}^{2^k-1} \left(\frac{1}{2^{L(n)}}\right)^{\alpha} = \sum_{j=0}^{k-1} \Big(\sum_{\substack{n \in \mathbb{N}\setminus\{0\}: \\ L(n)=j}} \left(\frac{1}{2^{L(n)}}\right)^{\alpha} \Big).$$

In the right-hand side of this last identity, for each $j \in \{0, 1, \ldots, k-1\}$, the inner summation runs over all natural numbers $n \ge 1$ such that $L(n) = j$. It follows that

$$\sum_{n=1}^{2^k-1} \left(\frac{1}{2^{L(n)}}\right)^{\alpha} = \sum_{j=0}^{k-1} \Big(\sum_{\substack{n \in \mathbb{N}\setminus\{0\}: \\ L(n)=j}} \left(\frac{1}{2^{L(n)}}\right)^{\alpha} \Big) = \sum_{j=0}^{k-1} \frac{1}{2^{j\alpha}} 2^j = \sum_{j=0}^{k-1} 2^{j(1-\alpha)} = \sum_{j=0}^{k-1} (2^{1-\alpha})^j.$$

If $\alpha \neq 1$, then $2^{1-\alpha} \neq 1$, and appealing again to the formula for the sum of the first k terms of a geometric series one gets that

$$\sum_{n=1}^{2^k-1} \left(\frac{1}{2^{L(n)}}\right)^{\alpha} = \sum_{j=0}^{k-1} (2^{1-\alpha})^j = \frac{1-(2^{1-\alpha})^k}{1-2^{1-\alpha}}.$$

If $0 \le \alpha < 1$, then $2^{1-\alpha} > 1$,

$$\lim_{k\to\infty} (2^{1-\alpha})^k = \infty$$

and

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^{L(n)}}\right)^{\alpha} = \lim_{k\to\infty} \sum_{n=1}^{2^k-1} \left(\frac{1}{2^{L(n)}}\right)^{\alpha} = \lim_{k\to\infty} \frac{1-(2^{1-\alpha})^k}{1-2^{1-\alpha}} = \infty.$$

If, on the other hand, $1 < \alpha < \infty$, then $2^{1-\alpha} < 1$ and

$$\lim_{k\to\infty} (2^{1-\alpha})^k = 0.$$

It follows that

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^{L(n)}}\right)^{\alpha} = \lim_{k\to\infty} \sum_{n=1}^{2^k-1} \left(\frac{1}{2^{L(n)}}\right)^{\alpha} = \lim_{k\to\infty} \frac{1-(2^{1-\alpha})^k}{1-2^{1-\alpha}} = \frac{1}{1-2^{1-\alpha}} < \infty.$$

Finally, if $\alpha = 1$, then

$$\sum_{n=1}^{2^k-1}\frac{1}{2^{L(n)}}=\sum_{j=0}^{k-1}(2^{1-1})^j=\sum_{j=0}^{k-1}1=k,$$

and so the series

$$\sum_{n=1}^{\infty}\frac{1}{2^{L(n)}}=\lim_{k\to\infty}\sum_{n=1}^{2^k-1}\frac{1}{2^{L(n)}}=\lim_{k\to\infty}k=\infty$$

again diverges.

(c) This time we start with the case $\alpha=1$. It follows from part (a) that

$$\sum_{n=1}^{\infty}\frac{1}{n}>\sum_{n=1}^{\infty}\frac{1}{2^{L(n)+1}}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2^{L(n)}}.$$

From the last paragraph of part (b) we know that this last series diverges, and so *the harmonic series diverges* as well:

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots=\infty.$$

If $0\le\alpha<1$, then parts (a) and (b) imply that

$$\sum_{n=1}^{\infty}\frac{1}{n^\alpha}>\sum_{n=1}^{\infty}\Big(\frac{1}{2^{L(n)+1}}\Big)^\alpha=\frac{1}{2^\alpha}\sum_{n=1}^{\infty}\Big(\frac{1}{2^{L(n)}}\Big)^\alpha=\infty,$$

and so the series with general term $n^{-\alpha}$ $(0\le\alpha<1)$ diverges as well.

Finally, if $1<\alpha<\infty$, then appealing again to parts (a) and (b), one has that

$$\sum_{n=1}^{\infty}\frac{1}{n^\alpha}\le\sum_{n=1}^{\infty}\Big(\frac{1}{2^{L(n)}}\Big)^\alpha=\frac{1}{1-2^{1-\alpha}}<\infty.$$

All in all, we conclude that the series

$$\sum_{n=1}^{\infty}\frac{1}{n^\alpha}$$

converges if and only if $1<\alpha<\infty$.

Problem 4 (Quasi Stirling formula)

(a) Let us start by proving that the sequence $\{\alpha_n\}$ is monotonically increasing. In light of Theorem 2.29, it will be enough to show that $\alpha_n\le\alpha_{n+1}$ for every $n\ge1$, i.e.

$$\Big(1+\frac{1}{n}\Big)^n\le\Big(1+\frac{1}{n+1}\Big)^{n+1}.$$

This follows from an elementary application of the AM-GM inequality (Problem 4, Übungsblatt 4) to the $n+1$ positive real numbers

$$x_1=1,\ \text{and}\ x_2=x_3=\ldots=x_{n+1}=1+\frac{1}{n},$$

for then

$$\Big(1+\frac{1}{n}\Big)^n=x_1x_2x_3\ldots x_{n+1}\le\Big(\frac{x_1+x_2+x_3+\ldots+x_{n+1}}{n+1}\Big)^{n+1}=\Big(\frac{1+n(1+\frac{1}{n})}{n+1}\Big)^{n+1}=\Big(1+\frac{1}{n+1}\Big)^{n+1},$$

as desired.

Let us now focus on the sequence $\{\beta_n\}$, for which we want to show

$$\Big(1+\frac{1}{n}\Big)^{n+1}\ge\Big(1+\frac{1}{n+1}\Big)^{n+2}$$

for every $n \geq 1$. We shall use Bernoulli's inequality which was already discussed in Präsenzübung 5. A variant thereof (whose easy proof runs by induction on m and will therefore be omitted) states that if $m \in \mathbb{N}$ and $x \in \mathbb{X}$,

$$(1+x)^m \geq 1 + mx.$$

Using this with $x = \frac{1}{n^2+2n}$ and $m = n+1$, one gets that

$$\frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+1}} = \left(1+\frac{1}{n^2+2n}\right)^{n+1} \geq 1 + \frac{n+1}{n^2+2n} \geq 1 + \frac{1}{n+1}.$$

It follows that

$$\left(1+\frac{1}{n}\right)^{n+1} \geq \left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+1}\right)^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+2},$$

as desired.

(b) Start by noting that the sequence α converges since it is monotonically increasing. This is the partial content of Theorem 3.5. It is also a bounded sequence since, for every natural number $n \geq 2$,

$$\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n}\right) = 1 - \frac{1}{n^2} \leq 1,$$

and so such numbers satisfy

$$\alpha_n = \left(1+\frac{1}{n}\right)^n \leq \left(1-\frac{1}{n}\right)^{-n} \leq \left(1-\frac{1}{2}\right)^{-2} = 4. \tag{9}$$

It follows that $\lim_{n\to\infty} \alpha_n$ is a (finite) nonnegative real number, which we will denote by e.

Now we relate the sequences α and β. Note that, for $n \geq 2$, the previous estimate (9) implies

$$\beta_n - \alpha_n = \left(1+\frac{1}{n}\right)^n \left(\left(1+\frac{1}{n}\right) - 1\right) \leq \frac{4}{n}.$$

Since $1 + 1/n \geq 1$, we further have that $\beta_n \geq \alpha_n$, and so

$$\alpha_n - \frac{4}{n} \leq \beta_n \leq \alpha_n + \frac{4}{n}$$

for $n \geq 2$. It follows that

$$\liminf_{n\to\infty} \alpha_n \leq \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \alpha_n.$$

Since we already know that $\liminf_{n\to\infty} \alpha_n = \limsup_{n\to\infty} \alpha_n = e$, it follows that

$$\liminf_{n\to\infty} \beta_n = \limsup_{n\to\infty} \beta_n,$$

and that the value of this common limit equals e.

Remark. It can be shown that $e = \sum_{n=0}^{\infty} \frac{1}{n!} \simeq 2.7182818284590452354...$

(c) Let us start by proving that, for every natural number $n \geq 2$,

$$\left(1+\frac{1}{1}\right)^1 \left(1+\frac{1}{2}\right)^2 \left(1+\frac{1}{3}\right)^3 \ldots \left(1+\frac{1}{n-1}\right)^{n-1} = \frac{n^n}{n!}.$$

We use induction on n, the base case $n = 2$ being trivial:

$$\left(1+\frac{1}{1}\right)^1 = 2 = \frac{2^2}{2!}.$$

Assuming the result holds for n, let us verify it for $n+1$:

$$\begin{aligned}\left(1+\frac{1}{1}\right)^1 \left(1+\frac{1}{2}\right)^2 \left(1+\frac{1}{3}\right)^3 \ldots \left(1+\frac{1}{n-1}\right)^{n-1}\left(1+\frac{1}{n}\right)^n &= \frac{n^n}{n!}\left(1+\frac{1}{n}\right)^n = \frac{\cancel{n^n}}{n!}\frac{(n+1)^n}{\cancel{n^n}} \\ &= \frac{(n+1)^n}{n!} = \frac{(n+1)^n}{n!}\frac{n+1}{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}.\end{aligned}$$

This proves the first identity.

The second one states that, for every natural number $n \geq 2$,

$$\left(1+\frac{1}{1}\right)^2\left(1+\frac{1}{2}\right)^3\left(1+\frac{1}{3}\right)^4\cdots\left(1+\frac{1}{n-1}\right)^n = \frac{n^n}{(n-1)!},$$

and we follow a similar route. The base case $n = 2$ amounts to checking that

$$\left(1+\frac{1}{1}\right)^2 = 4 = \frac{2^2}{1!}.$$

Assuming that the statement has been verified for n, let us check it for $n+1$:

$$\begin{aligned}\left(1+\frac{1}{1}\right)^2\left(1+\frac{1}{2}\right)^3\left(1+\frac{1}{3}\right)^4\cdots\left(1+\frac{1}{n-1}\right)^n\left(1+\frac{1}{n}\right)^{n+1} &= \frac{n^n}{(n-1)!}\left(1+\frac{1}{n}\right)^{n+1} = \frac{n^n}{(n-1)!}\frac{(n+1)^{n+1}}{n^{n+1}} \\ &= \frac{\cancel{n^n}}{(n-1)!}\frac{(n+1)^{n+1}}{n\cdot\cancel{n^n}} = \frac{(n+1)^{n+1}}{n!}.\end{aligned}$$

This proves the second identity.

(d) The first identity proved in part (c) translates into

$$\alpha_1\alpha_2\alpha_3\ldots\alpha_{n-1} = \frac{n^n}{n!}, \tag{10}$$

whereas the second one translates into

$$\beta_1\beta_2\beta_3\ldots\beta_{n-1} = \frac{n^n}{(n-1)!}. \tag{11}$$

Since the sequence α is monotonically increasing, we have that

$$\alpha_n \leq \lim_{n\to\infty}\alpha_n = e, \quad \text{for every } n \in \mathbb{N}\setminus\{0\}.$$

Similarly, since the sequence β is monotonically decreasing, we have that

$$\lim_{n\to\infty}\beta_n = e \leq \beta_n, \quad \text{for every } n \in \mathbb{N}\setminus\{0\}.$$

It follows that

$$\alpha_1\alpha_2\alpha_3\ldots\alpha_{n-1} \leq e^{n-1} \leq \beta_1\beta_2\beta_3\ldots\beta_{n-1},$$

and so (10) and (11) imply that

$$\frac{n^n}{n!} \leq e^{n-1} \leq \frac{n^n}{(n-1)!}.$$

These inequalities can be rewritten as

$$e\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n,$$

and we are done.