Analysis 1, Solutions to problem set 3

Problem 1 (Binomial coefficients)

(a) Note that, for all the binomial coefficients below to be well-defined, we need $k < n$. Using the definition of the binomial coefficients, the definition of factorial, and elementary arithmetic properties of rational numbers, we compute:

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-(k+1))!} \\
&= \frac{n!}{k!(n-k)(n-(k+1))!} + \frac{n!}{(k+1)k!(n-(k+1))!} \\
&= \frac{n!}{k!(n-(k+1))!}\Big(\frac{1}{n-k} + \frac{1}{k+1}\Big) \\
&= \frac{n!}{k!(n-(k+1))!} \cdot \frac{(\cancel{k}+1)+(n-\cancel{k})}{(n-k)(k+1)} \\
&= \frac{(n+1)n!}{(k+1)k!(n-k)(n-(k+1))!} \\
&= \frac{(n+1)!}{(k+1)!(n-k)!} \\
&= \binom{n+1}{k+1}.
\end{aligned}$$

(b) Let k and n be natural numbers. Let us use induction on n to show that

$$\binom{n}{k} \text{ is a natural number, for every } k \leq n.$$

The base case $n = 0$ is easy. If k is a natural number and $k \leq n = 0$, then $k = 0$, and

$$\binom{0}{0} = \frac{0!}{0! \cdot (0-0)!} = 1$$

is indeed a natural number. Assuming that the statement has been verified for n, let us prove that

$$\binom{n+1}{k} \text{ is a natural number, for every } k \leq n+1.$$

From part (a) we know that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}. \tag{1}$$

By induction hypothesis, both of the summands in the right-hand side of this last identity are natural numbers. The sum of two natural numbers is still a natural number, and so is $\binom{n+1}{k}$, as we wanted to show. The proof is almost complete, except for the following caveats: if k is a natural number, then $k-1$ is also a natural number *only if* $k \neq 0$; if $k \leq n+1$, then $k \leq n$ *only if* $k \neq n+1$. In particular, identity (1) is ill-defined for $k = 0$ or $k = n+1$. In these cases however, it is straightforward to verify that

$$\binom{n+1}{0} = \binom{n+1}{n+1} = 1,$$

again a natural number. The proof is now complete.

Alternative solution. This time we will show more, namely that the number of k-element subsets of an n-element set, say $X_n := \{x_1, x_2, \ldots x_n\}$, is precisely $\binom{n}{k}$. It follows that the numbers $\binom{n}{k}$ are indeed integers, a fact that might not be immediately apparent from the definition of the binomial coefficients given in the statement of the problem.

Let us start by addressing the base case, $n = 0$. The only set with zero elements is the empty set, which is its own unique subset. On the other hand, $\binom{0}{0} = 1$, as we have already seen.

Let us assume that the claim has been verified for all subsets of X_n, and consider an $(n+1)$-element set $X_{n+1} := \{x_1, x_2, \ldots, x_n, x_{n+1}\}$. The claim is trivial for $k = 0$ and $k = n+1$, and so we will restrict attention to $1 \leq k \leq n$. The elementary but crucial combinatorial observation is the following: *each k-element subset of X_{n+1} either contains the element x_{n+1}, or it does not.* The number of subsets which fall into the latter category equals $\binom{n}{k}$, *by induction hypothesis.* The number of subsets which fall into the former category equals $\binom{n}{k-1}$, also by induction hypothesis; this holds because one of their elements (namely, x_{n+1}) is fixed to begin with. It follows that the total number of k-element subsets of an $(n+1)$-element set is

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k},$$

where the identity is a consequence of part (a). This completes the induction step and proves the claim.

(c) The base case $n = 0$ is easy to verify:

$$(x+y)^0 = 1 = \binom{0}{0} x^{0-0} y^0 = \sum_{k=0}^{0} \binom{0}{k} x^{0-k} y^k.$$

Let us assume that the case n has already been verified. Then:

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)^n (x+y) \\
&= \Big(\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \Big) (x+y) \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1} \\
&= x^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum_{k=0}^{n-1} \binom{n}{k} x^{n-k} y^{k+1} + y^{n+1} \\
&= x^{n+1} + \sum_{j=0}^{n-1} \binom{n}{j+1} x^{n-(j+\cancel{1}-\cancel{1})} y^{j+1} + \sum_{k=0}^{n-1} \binom{n}{k} x^{n-k} y^{k+1} + y^{n+1} \\
&= x^{n+1} + \sum_{j=0}^{n-1} \left(\binom{n}{j+1} + \binom{n}{j} \right) x^{n-j} y^{j+1} + y^{n+1} \\
&\overset{(a)}{=} x^{n+1} + \sum_{j=0}^{n-1} \binom{n+1}{j+1} x^{n-j} y^{j+1} + y^{n+1} \\
&= x^{n+1} + \sum_{j=0}^{n} \binom{n+1}{j+1} x^{n-j} y^{j+1} \\
&= x^{n+1} + \sum_{k=1}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k.
\end{aligned}$$

Problem 2 (Inequalities)

(a) We start by making use of the binomial formula from problem 1(c) to expand:

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}. \tag{2}$$

Since all the summands in this sum are positive numbers, we can estimate the whole sum *from below* by its first two summands: [1]

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \geq \binom{n}{0}\frac{1}{n^0} + \binom{n}{1}\frac{1}{n^1} = 1\frac{1}{1} + n\frac{1}{n} = 1+1 = 2.$$

In view of identity (2) it follows that, for every natural number $n \geq 1$,

$$\left(1+\frac{1}{n}\right)^n \geq 2.$$

(b) Start by noting that the statement

$$n! \leq \left(\frac{n}{2}\right)^n$$

fails for $n \in \{1,2,3,4,5\}$. This is the result of a straightforward computation which we will not present here. Let us prove by induction that

$$\forall n \in \mathbb{N} : n \geq 6 \Rightarrow n! \leq \left(\frac{n}{2}\right)^n. \tag{3}$$

The base case is easy to verify:

$$6! = 720 \leq 729 = 3^6 = \left(\frac{6}{2}\right)^6.$$

Assuming that the statement has been verified for n, let us prove it for $n+1$. Using the definition of factorial and the induction hypothesis,

$$(n+1)! = (n+1)n! \leq (n+1)\left(\frac{n}{2}\right)^n.$$

On the other hand, the right-hand side of this inequality is controlled by the desired quantity, i.e.,

$$(n+1)\left(\frac{n}{2}\right)^n \leq \left(\frac{n+1}{2}\right)^{n+1}$$

if and only if

$$2n^n \leq (n+1)^n$$

if and only if

$$2 \leq \left(1+\frac{1}{n}\right)^n.$$

This was the content of part (a) and so we are done. This concludes the induction argument and establishes (3).

(c) Let us first prove that

$$\frac{1}{n^k}\binom{n}{k} \leq \frac{1}{k!} \tag{4}$$

for every natural $n \geq 1$ and $k \in \{2,3,\dots,n\}$. Recalling the definition of the binomial coefficient $\binom{n}{k}$, we see that inequality (4) is equivalent to

$$\frac{1}{n^k}\frac{n!}{k!(n-k)!} \leq \frac{1}{k!},$$

which in turns holds if and only if

$$\frac{n!}{(n-k)!} \leq n^k.$$

Notice that the left-hand side of this last inequality equals the product of the k natural numbers $n-k+1, n-k+2, \dots, n-1, n$. All the following k inequalities

[1] Note carefully that these are well defined since $n \geq 1$.

$$n - j \le n \text{ for every } j \in \{0, 1, \ldots, k-1\}$$

hold trivially, and can be multiplied together to yield the desired result.

The same idea allows us to prove that

$$\frac{1}{k!} \le \frac{1}{2^{k-1}}.$$

In fact, this holds if and only if

$$2^{k-1} \le k!, \tag{5}$$

and

$$2 \le k - j \text{ for every natural } k \ge 2 \text{ and } j \in \{0, 1, \ldots, k-2\}.$$

Multiplying together these $k-1$ inequalities establishes (5).

Remark. Induction is still needed for a complete proof of either of these inequalities, but to avoid repetitive arguments only the intuitive "proof" is outlined here. The dedicated reader is in any case urged to provide a formal proof.

(d) The statement is trivial to verify for $n = 1$, so let us assume $n \ge 2$. Recall formula (2) from part (a), and use inequality (4) from part (c) to estimate:

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \\
&= \binom{n}{0} \frac{1}{n^0} + \binom{n}{1} \frac{1}{n^1} + \sum_{k=2}^{n} \binom{n}{k} \frac{1}{n^k} \\
&\le 1 + 1 + \sum_{k=2}^{n} \frac{1}{2^{k-1}}. \qquad (6)
\end{aligned}$$

This last sum can be computed exactly via the formula for a geometric sum derived in Problem 2(c) from last week's homework assignment. For $n \ge 2$, we get that

$$\sum_{k=2}^{n} \frac{1}{2^{k-1}} = \frac{1}{2} \sum_{k=0}^{n-2} \left(\frac{1}{2}\right)^k = \frac{1}{2} \frac{1 - (1/2)^{n-1}}{1 - 1/2} = 1 - \left(\frac{1}{2}\right)^{n-1} \le 1.$$

From this and estimate (6) it finally follows that

$$\left(1 + \frac{1}{n}\right)^n \le 1 + 1 + \sum_{k=2}^{n} \frac{1}{2^{k-1}} \le 1 + 1 + 1 = 3.$$

(e) Let us use induction to prove that

$$\forall n \in \mathbb{N} : n \ge 1 \Rightarrow n! \ge \left(\frac{n}{3}\right)^n. \tag{7}$$

The base case $n = 1$ is immediate, for

$$1! = 1 \ge \frac{1}{3} = \left(\frac{1}{3}\right)^1.$$

Assuming that the statement is valid for n, let us establish it for $n + 1$. Using the definition of factorial and the induction hypothesis, we have that

$$(n+1)! = (n+1)n! \ge (n+1)\left(\frac{n}{3}\right)^n.$$

Now,

$$(n+1)\left(\frac{n}{3}\right)^n \ge \left(\frac{n+1}{3}\right)^{n+1}$$

if and only if

$$3n^n \ge (n+1)^n$$

if and only if

$$3 \ge \left(1 + \frac{1}{n}\right)^n.$$

This was the content of part (d) and we are done. The induction argument is complete and (7) follows.

Problem 3 (Peano axioms, revisited)

(a) Let us start by establishing the statement

$$\forall x \exists y \exists z \forall w : \Big(y = x \wedge x = 0\Big) \vee \Big(y = x \wedge \nu(z) = x\Big) \vee \Big(y \neq x \wedge x \neq 0 \wedge \nu(w) \neq x\Big), \tag{8}$$

which loosely speaking states that any object x is either 0, or the successor of a natural number, or anything else; moreover, in the first two cases we may choose y satisfying $y = x$, and in the third case we may choose y satisfying $y \neq x$.

For the purpose of establishing (8), let x be arbitrary. The statements $\exists z : \nu(z) = x$ and $\forall w : \nu(w) \neq x$ are dual to each other, and so one of them must hold by TND. In other words, given x,

$$\exists z \forall w : \nu(z) = x \vee \nu(w) \neq x.$$

The order of the quantifiers in this last statement can be reversed in view of Theorem 1.15 (3). Thus, given x and w, there exists z such that

$$\nu(z) = x \vee \nu(w) \neq x.$$

By TND we also know $x = 0 \vee x \neq 0$, and so

$$(x = 0 \vee x \neq 0) \wedge (\nu(z) = x \vee \nu(w) \neq x),$$

where the variables x, w, z are quantified as before. By distributivity and a further application of TND,

$$(x = 0) \vee (x \neq 0 \wedge \nu(z) = x) \vee (x \neq 0 \wedge \nu(w) \neq x).$$

We are thus led to considering three distinct cases. Setting $y = x$ in the first case, $y = x$ in the second case and $y \neq x$ in the third case (this is possible in view of Rules 18 and 19 from the Skript), we conclude that

$$\forall x \exists y \exists z \forall w : \Big(y = x \wedge x = 0\Big) \vee \Big(y = x \wedge x \neq 0 \wedge \nu(z) = x\Big) \vee \Big(y \neq x \wedge x \neq 0 \wedge \nu(w) \neq x\Big).$$

That this implies (8) is a straightforward consequence of axiom (P2).

Having verified statement (8), we are now ready to use the axiom of choice (AC, Rule 34) on the statement $\forall x \exists y : P(x, y)$, where $P(x, y)$ is defined to be the statement

$$\exists z \forall w : \Big(y = x \wedge x = 0\Big) \vee \Big(y = x \wedge \nu(z) = x\Big) \vee \Big(y \neq x \wedge x \neq 0 \wedge \nu(w) \neq x\Big). \tag{9}$$

We conclude the existence of a function g such that, for every x,

$$\Big(P(x, g(x)) \wedge g(x) \neq g \wedge \nu(x) \neq \nu\Big) \vee \Big(g(x) = g \wedge \nu(x) = \nu\Big).$$

Plugging in the statement (9) for P, we see that the function g just constructed satisfies, for every x,

$$\exists z \forall w : \tag{10}$$
$$\Big(\Big((g(x) = x \wedge x = 0) \vee (g(x) = x \wedge \nu(z) = x) \vee (g(x) \neq x \wedge x \neq 0 \wedge \nu(w) \neq x)\Big) \wedge g(x) \neq g \wedge \nu(x) \neq \nu\Big) \vee (g(x) = g \wedge$$

We now apply the specification axiom (Rule 35) to the function g. It follows that there exists a function f such that

$$\forall x : (f(x) = f \wedge g = x) \vee (f(x) \neq f \wedge g(x) = x \wedge g \neq x) \vee (f(x) = f \wedge g(x) \neq x \wedge g \neq x). \tag{11}$$

We claim that f satisfies the property required in the statement of the problem. To check this, start by recalling that x is in the domain of f if and only if $f(x) \neq f$. By (11), this happens only if $g(x) = x$ and $g \neq x$, and therefore only if $g(x) \neq g$. By (10), this implies that $\nu(x) \neq \nu$ and that

$$\exists z \forall w : (g(x) = x \wedge x = 0) \vee (g(x) = x \wedge \nu(z) = x) \vee (g(x) \neq x \wedge x \neq 0 \wedge \nu(w) \neq x).$$

Since $g(x) = x$, it follows that $x = 0$ or $\nu(z) = x$ for some z. Since $\nu(x) \neq \nu$, we have that $x \neq \nu$, and so z is in the domain of ν. In other words, z is a natural number. In particular, x is either 0 or the successor of a natural number.

Conversely, let x be either 0 or the successor of a natural number. Let us check that x is in the domain of f:

- If $x = 0$, then we claim that $f(0) \neq f$. Since $\nu(0) \neq \nu$, it follows from (10) that $g(0) \neq g$ and that $g(0) = 0$. In particular, $g \neq 0$ and so from (11) it follows that $f(0) \neq f$, as claimed.
- If $x = \nu(z)$ for some natural number z, we claim that $f(x) \neq f$. Since $x = \nu(z)$ for some natural number z, it follows from axiom (P4) that $\nu(x) \neq \nu$. Since $\nu(x) \neq \nu$, it follows from (10) that $g(x) \neq g$ and that $g(x) = x$. In particular, $g \neq x$ and so from (11) it follows that $f(x) \neq f$, as claimed.

In both cases we concluded that $f(x) \neq f$, and this finishes the proof.

(b) The strategy will be to show that the hypothesis of (P5) is satisfied for the function f constructed in part (a), and use the conclusion of (P5) to derive the result. We have to show that

(i) $f(0) \neq f$;

(ii) $\forall x \forall y : \nu(x) = \nu \vee f(x) = f \vee \nu(x) \neq y \vee f(y) \neq f$.

Part (i) was already shown in the course of part (a).

For part (ii), take x, y such that $\nu(x) \neq \nu$ and $f(x) \neq f$ (i.e. x is in the domains of ν and f), and $\nu(x) = y$ (i.e. y is the successor of x). Our mission is to show that $f(y) \neq f$, i.e. that $y = \nu(x)$ is in the domain of f. This was also already done in part (a).

Having verified (i) and (ii), we can now apply (P5) to conclude that

$$\forall x : f(x) \neq f \vee \nu(x) = \nu.$$

This means that an arbitrary given x is either in the domain of f or is not in the domain of ν. In particular, the domain of ν is contained in the domain of f, as required.

From part (b) it follows that every element of the domain of ν (i.e. every natural number) is contained in the domain of f. From part (a) we know that the domain of f consists precisely of those natural numbers which are either 0, or successors of other natural numbers. It follows that every nonzero natural number is a successor of a(nother) natural number. QED.

Problem 4 (Identity functions)

We start by noting that property (I4) implies that the domains of φ and $\widetilde{\varphi}$ are both contained in the domain of ν, which by definition equals $\mathbb{N}$.
It will be enough to show that

$$\forall x : (\varphi(x) = \widetilde{\varphi}(x) \wedge \varphi(x) \neq \varphi \wedge \widetilde{\varphi}(x) \neq \widetilde{\varphi}) \vee (\varphi(x) = \varphi \wedge \widetilde{\varphi}(x) = \widetilde{\varphi}),$$

for the desired claim $\varphi = \widetilde{\varphi}$ will then follow via an application of the extension axiom (Rule 38).
Let us take x such that $\varphi(x) \neq \varphi$ *or* $\widetilde{\varphi}(x) \neq \widetilde{\varphi}$. Without loss of generality, assume that the first case holds i.e. $\varphi(x) \neq \varphi$. We want to show that $\widetilde{\varphi}(x) \neq \widetilde{\varphi}$ and that $\varphi(x) = \widetilde{\varphi}(x)$.
Step 1. Let us prove that $\widetilde{\varphi}(x) \neq \widetilde{\varphi}$.
Initial remark. In particular, this will show that the domain of φ is contained in the domain of $\widetilde{\varphi}$. A symmetric argument shows the reverse inclusion, and so it follows that the functions φ and $\widetilde{\varphi}$ have the same domain.
Preliminaries. Let us start by considering the following statement, which holds in light of Rules 18 and 19:

$$\forall x \exists y : \begin{cases} y = x & \text{if } \varphi(x) \neq \varphi \\ y \neq x & \text{if } \varphi(x) = \varphi \end{cases}$$

Applying the axiom of choice to this statement, we conclude the existence of a function h for which the following property, which we denote by (AC_h), holds

$$\forall x : \left[\left\{ \begin{array}{ll} h(x) = x & \text{if } \varphi(x) \neq \varphi \\ h(x) \neq x & \text{if } \varphi(x) = \varphi \end{array} \right\} \wedge h(x) \neq h \wedge \nu(x) \neq \nu \right] \vee [h(x) = h \wedge \nu(x) = \nu]$$

Considering $\widetilde{\varphi}$ instead of φ, we similarly conclude the existence of a function $\widetilde{h}$ with the corresponding property $(AC_{\widetilde{h}})$.
Main argument. Since $\varphi(x) \neq \varphi$, it follows by (I4) that $\nu(x) \neq \nu$ (and so x is a natural number). Then (AC_h) implies $h(x) = x$. We will make use of the following lemma, whose proof is deferred to the end:

Lemma 1. $\forall x \in \mathbb{N} : h(x) = x \Rightarrow \widetilde{h}(x) = x$.

Since $h(x) = x$ (and x is a natural number), it follows from Lemma 1 that $\widetilde{h}(x) = x$. But then $(AC_{\widetilde{h}})$ implies $\widetilde{\varphi}(x) \neq \widetilde{\varphi}$, as desired. This concludes Step 1.
Step 2. Let us now prove that $\varphi(x) = \widetilde{\varphi}(x)$. We are working under the assumption that $\varphi(x) \neq \varphi$. By Step 1, we also know that $\widetilde{\varphi}(x) \neq \widetilde{\varphi}$. Applying (I1) twice, we conclude that $\varphi(x) = x$ and that $\widetilde{\varphi}(x) = x$. In particular (use substitution), $\varphi(x) = \widetilde{\varphi}(x)$, as desired. This concludes Step 2, and finishes the proof modulo the verification of Lemma 1.

Proof of Lemma 1. We shall use induction on x.
The base case $x = 0$ is easy to verify. From (I2) we know that $\varphi(0) \neq \varphi$. Since moreover $\nu(0) \neq \nu$, we conclude from (AC_h) that $h(0) = 0$. A parallel argument that uses $(AC_{\widetilde{h}})$ instead of (AC_h) shows that $\widetilde{h}(0) = 0$, and so the claim holds.
Let us assume that x is a natural number for which the statement

$$h(x) = x \Rightarrow \widetilde{h}(x) = x$$

holds. This is our induction hypothesis. Using it, let us verify that

$$h(\nu(x)) = \nu(x) \Rightarrow \widetilde{h}(\nu(x)) = \nu(x).$$

Let us then assume that $\boxed{h(\nu(x)) = \nu(x)}$. By $(AC_{\widetilde{h}})$, it is enough to show that $\widetilde{\varphi}(\nu(x)) \neq \widetilde{\varphi}$ (since $\nu(\nu(x)) \neq \nu$). Let us split the analysis in two cases:

Case 1. If $\nu(x) = N$, then $\varphi(\nu(x)) = \varphi$ in light of (I3). This contradicts the assumption $h(\nu(x)) = \nu(x)$ under which we are working, for then $\varphi(\nu(x)) \neq \varphi$ by (AC_h). It follows that Case 1 cannot arise.

Case 2. If $\nu(x) \neq N$, we claim that $\widetilde{\varphi}(x) \neq \widetilde{\varphi}$. If so, it follows from (I5) that $\widetilde{\varphi}(\nu(x)) \neq \widetilde{\varphi}$, and we are done. Therefore we just need to establish the claim. Aiming at a contradiction, assume it does not hold. This means that $\widetilde{\varphi}(x) = \widetilde{\varphi}$. By $(AC_{\widetilde{h}})$, it follows that $\widetilde{h}(x) \neq x$ and so, *by induction hypothesis*, $h(x) \neq x$. But then (AC_h) implies that $\varphi(x) = \varphi$, and so $\varphi(\nu(x)) = \varphi$ by (I6). Again by (AC_h), it follows that $h(\nu(x)) \neq \nu(x)$, a contradiction to our boxed assumption. This establishes the claim and finishes the analysis of Case 2.

This completes the proof of the lemma. $\square$

Final remarks. (i) Notice that all six properties (I1)-(I6) were used at least once. (ii) It turns out that if $N = 0$, no function satisfying (I1)-(I6) exists; already from (I2) and (I3) we would get a contradiction. However, uniqueness is trivially guaranteed in this case. (iii) Contrary to Problem 3, we did not need to invoke the axiom of specification. However, that will be crucial for the existence proof which was left for Problem 2 in Präsenzsübung.