Analysis 1, Solutions to problem set 2

Problem 1 (Quantifiers)

(a) We take $\forall x \exists y : P(y) \wedge Q(x)$ as our starting point, and let x' be a variable. Then Rule 21 implies $\exists y : P(y) \wedge Q(x')$. Now, let y' be a variable. Then Rule 17 implies $P(y') \wedge Q(x')$, and then Rule 11 implies that $P(y')$ and $Q(x')$ both hold. From Rule 20, it then follows that $\forall x' : Q(x')$. This, together with $P(y')$, implies that $\forall x' : Q(x') \wedge P(y')$ via an argument entirely parallel to that used to prove Theorem 1.14. *This is where we use the assumption that the statement P does not contain the variable x to begin with.* It follows that $\forall x' : P(y') \wedge Q(x')$. To see why this is the case, introduce a new variable x'' and apply Rule 21 to the statement $\forall x' : Q(x') \wedge P(y')$ to derive $Q(x'') \wedge P(y')$. Then use the symmetry of $\wedge$ to infer $P(y') \wedge Q(x'')$, and then introduce one last variable x''' and apply Rule 20 to conclude $\forall x''' : P(y') \wedge Q(x''')$, which is trivially equivalent to $\forall x' : P(y') \wedge Q(x')$. Finally, $\exists y' \forall x' : P(y') \wedge Q(x')$ follows from this by a final application of Rule 16. This is of course equivalent to what we wanted to prove, $\exists y \forall x : P(y) \wedge Q(x)$.

(b) We take $\forall x \exists y : P(y) \vee Q(x)$ as our starting point, and let x' be a variable. Then Rule 21 implies $\exists y : P(y) \vee Q(x')$. Now, let y' be a variable. Then Rule 17 implies $P(y') \vee Q(x')$, and then Rule 9 implies that (i) $P(y')$ holds, or (ii) $Q(x')$ holds. We are therefore led to consider two distinct cases:

Case (i) From $P(y')$ we may conclude that $\forall x' : Q(x') \vee P(y')$ by an application of Theorem 1.14. As in part (a), the symmetry of $\vee$ allows us to infer from this that $\forall x' : P(y') \vee Q(x')$, and then $\exists y' \forall x' : P(y') \vee Q(x')$ follows by a final application of Rule 16.

Case (ii) From $Q(x')$ we conclude $P(y') \vee Q(x')$ from Rule 8. We then have that $\forall x' : P(y') \vee Q(x')$ by an application of Rule 20, and then $\exists y' \forall x' : P(y') \vee Q(x')$ follows from this by a final application of Rule 16 as in Case (i) or part (a).

In both cases we conclude that $\exists y' \forall x' : P(y') \vee Q(x')$, and so we can unconditionally conclude $\exists y' \forall x' : P(y') \vee Q(x')$ via an application of Rule 9. This is of course equivalent to what we wanted to prove, $\exists y \forall x : P(y) \vee Q(x)$.

(c) Let P and Q both be the statement $x = y$. Note that both P and Q depend on *both* variables x and y. Then $P \vee Q$ implies both P and Q (since they are the same statement), and the statement

$$\forall x \exists y : x = y$$

is *true*. It claims that no matter how we assign an object to x, we may find one object which we can assign to y so that $x = y$, and it follows from Rule 18. On the other hand, the statement

$$\exists y \forall x : x = y$$

is in general *false*. We cannot find an object to assign to y so that no matter what we assign to x we have the stated equality, unless our universe consists of exactly one single object. In greater detail: assume $\exists y \forall x : x = y$. Choose y' such that $\forall x : x = y'$. From Rule 19 we know that $\exists x : y' \neq x$. Choose x' such that $y' \neq x'$. Since $\forall x : x = y'$, we have in particular that $x' = y'$, and so $y' = x'$. But $(y' \neq x') \wedge (y' = x')$ leads to an explosion via *ex falso quodlibet*, and we are done.

Remark. (Thanks to Kevin Wilkinghoff for pointing this out) Strictly speaking, this solution only takes care of statements P, Q of order 0, an assumption which is *not* present in the statement of the problem. For higher order statements, one would need to appeal to Rules 23-26 instead. The changes are straightforward and are left to the interested reader.

Problem 2 (Induction)

(a) We start by verifying the base case $n = 0$:

$$\sum_{k=0}^{0} k = 0 = \frac{0(0+1)}{2}.$$

Let us assume that the statement has been established for a generic n. Let us prove it for $n+1$:

$$\sum_{k=0}^{n+1} k = \Big(\sum_{k=0}^{n} k\Big) + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}.$$

Here, the first equality follows from the recursive definition of the symbol $\sum_{k=0}^{n+1}$, the second equality follows from the induction hypothesis, and the third equality follows from the distributive property of multiplication with respect to addition.

(b) The base case $n = 0$ is again easy to verify:

$$\sum_{k=0}^{0} k^3 = 0^3 = 0 = 0^2 = \Big(\sum_{k=0}^{0} k\Big)^2.$$

Assuming that the result is known to hold for a certain n, let us derive it for $n+1$:

$$\sum_{k=0}^{n+1} k^3 = \sum_{k=0}^{n} k^3 + (n+1)^3 = \Big(\sum_{k=0}^{n} k\Big)^2 + (n+1)^3 = \Big(\frac{n(n+1)}{2}\Big)^2 + (n+1)^3 = (n+1)^2\Big(\frac{n^2}{4} + n + 1\Big) = \Big(\frac{(n+1)(n+2)}{2}\Big)^2.$$

Here, the first equality follows from the recursive definition of the symbol $\sum_{k=0}^{n+1}$, the second equality follows from the induction hypothesis, the third equality follows from part (a), and the fourth and fifth equalities follow from elementary arithmetic properties of rational numbers.

(c) The base case $n = 0$ runs as follows: since $x \neq 1$,

$$\sum_{k=0}^{0} x^k = x^0 = 1 = \frac{1-x}{1-x} = \frac{1-x^{0+1}}{1-x}.$$

Assuming we have the result for n, let us establish it for $n+1$:

$$\sum_{k=0}^{n+1} x^k = \sum_{k=0}^{n} x^k + x^{n+1} = \frac{1-x^{n+1}}{1-x} + x^{n+1} = \frac{1-x^{n+1}+x^{n+1}(1-x)}{1-x} = \frac{1-x^{n+1}+x^{n+1}-x^{n+2}}{1-x} = \frac{1-x^{n+2}}{1-x}.$$

The first and second equalities follow as in parts (a) or (b), whereas the third, fourth and fifth equalities follows from elementary arithmetic properties of real numbers.

(d) As usual, the base case $n = 0$ is easy enough:

$$\Big(\sum_{k=0}^{0} x_k y_k\Big)^2 = (x_0 y_0)^2 = x_0^2 y_0^2 = \Big(\sum_{k=0}^{0} x_k^2\Big)\Big(\sum_{k=0}^{0} y_k^2\Big).$$

The inductive step reads as follows:

$$\begin{aligned}
\Big(\sum_{k=0}^{n+1} x_k y_k\Big)^2 &\overset{(1)}{=} \Big(\sum_{k=0}^{n} x_k y_k + x_{n+1} y_{n+1}\Big)^2 \overset{(2)}{=} \Big(\sum_{k=0}^{n} x_k y_k\Big)^2 + 2x_{n+1}y_{n+1}\sum_{k=0}^{n} x_k y_k + x_{n+1}^2 y_{n+1}^2 \\
&\overset{(3)}{\leq} \Big(\sum_{k=0}^{n} x_k^2\Big)\Big(\sum_{k=0}^{n} y_k^2\Big) + 2x_{n+1}y_{n+1}\sqrt{\Big(\sum_{k=0}^{n} x_k^2\Big)\Big(\sum_{k=0}^{n} y_k^2\Big)} + x_{n+1}^2 y_{n+1}^2 \\
&\overset{(4)}{=} \Big(\sum_{k=0}^{n} x_k^2\Big)\Big(\sum_{k=0}^{n} y_k^2\Big) + 2\Big(x_{n+1}\sqrt{\sum_{k=0}^{n} y_k^2}\Big)\Big(y_{n+1}\sqrt{\sum_{k=0}^{n} x_k^2}\Big) + x_{n+1}^2 y_{n+1}^2 \\
&\overset{(5)}{\leq} \Big(\sum_{k=0}^{n} x_k^2\Big)\Big(\sum_{k=0}^{n} y_k^2\Big) + \Big(x_{n+1}^2 \sum_{k=0}^{n} y_k^2 + y_{n+1}^2 \sum_{k=0}^{n} x_k^2\Big) + x_{n+1}^2 y_{n+1}^2 \\
&\overset{(6)}{=} \Big(\sum_{k=0}^{n} x_k^2 + x_{n+1}^2\Big)\Big(\sum_{k=0}^{n} y_k^2 + y_{n+1}^2\Big) \\
&\overset{(7)}{=} \Big(\sum_{k=0}^{n+1} x_k^2\Big)\Big(\sum_{k=0}^{n+1} y_k^2\Big).
\end{aligned}$$

Here, (1) follows from the recursive definition of the symbol $\sum_{k=0}^{n+1}$ and (2) follows from expanding the square. Inequality (3) follows from two separate applications of the induction hypothesis, the second of which is used together with extraction of square roots on both sides. This is where we use the hypotheses that $x_i, y_i \geq 0$ for every $i \in \{0, 1, \ldots, n\}$. Equality (4) is just a consequence of commutativity of multiplication together with the fact that $\sqrt{\alpha\beta} = \sqrt{\alpha}\sqrt{\beta}$ for arbitrary real numbers $\alpha, \beta \geq 0$. Inequality (5) amounts to two applications of a result which we already established in Übungsblatt 1, Problem 3 (d): for arbitrary real numbers $\alpha, \beta \geq 0$, one has that

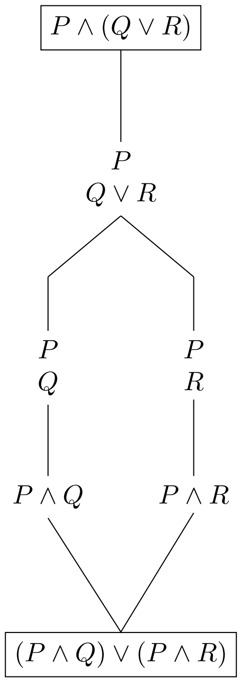
$$2\alpha\beta \leq \alpha^2 + \beta^2. \tag{1}$$

Recall that this is nothing but a reformulation of the trivial fact that $(\alpha - \beta)^2 \geq 0$. Inequality (1) is first applied with $\alpha = x_{n+1}$ and $\beta = \sqrt{\sum_{k=0}^{n} y_k^2}$, and then with $\alpha = y_{n+1}$ and $\beta = \sqrt{\sum_{k=0}^{n} x_k^2}$. Coincidentally, the case $n = 1$ of the Cauchy-Schwarz inequality which we are trying to prove amounts to inequality (1). Last but not least, identity (6) follows from the distributive property of multiplication with respect to addition together with the commutativity of addition and multiplication, whereas (7), just like (1), follows from the recursive definition of the symbol $\sum_{k=0}^{n+1}$.

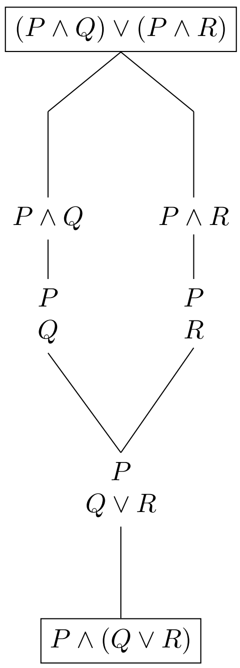
Problem 3 (Distributive laws)

We proceed similarly to the proof of Theorem 1.12 from the lecture notes:

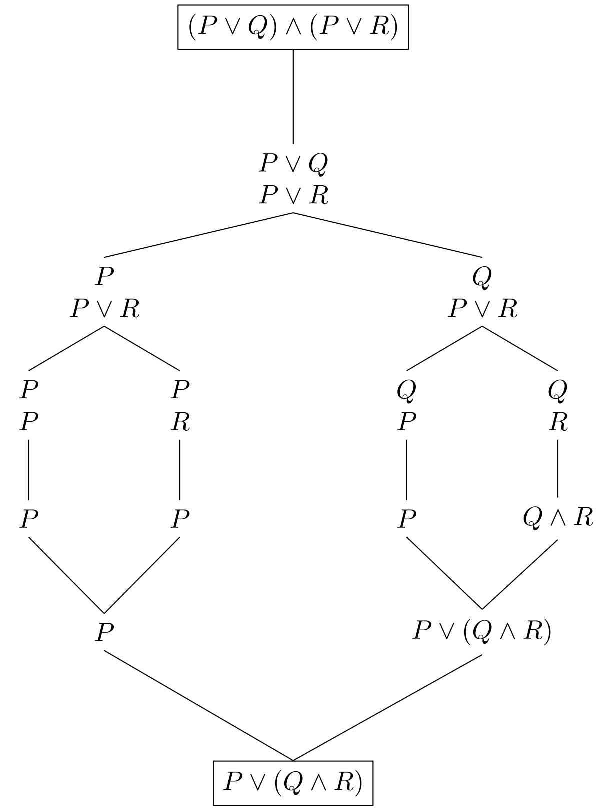
(a) .



(b) .



(c) .



Problem 4 (Parity)

As usual, let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers, which *we define in such a way that* $0 \in \mathbb{N}$.

(a) Consider the set

$$X := \{n \in \mathbb{N} : n \text{ is even or } n \text{ is odd}\}.$$

We want to show that $X = \mathbb{N}$. Clearly, $X \subseteq \mathbb{N}$, and so our task is reduced to showing that $\mathbb{N} \subseteq X$. This will follow from axiom (P5), provided we show that:

(i) $0 \in X$;

(ii) $\forall n \in \mathbb{N} : (n \in X \Rightarrow n + 1 \in X)$.

For part (i), notice that 0 can be written in the form $0 = a + a$ for some natural number $a \in \mathbb{N}$. Indeed, it is enough to take $a = 0$, for $0 \in \mathbb{N}$ by axiom (P1), and $0 + 0 = 0$. This means that, *by definition*, 0 is an even number.

For part (ii), consider any $n \in X$. By construction of the set X, this means that n is even or n is odd. We are thus led to splitting the analysis into two cases:

Case 1. *n is even.* By definition of evenness, there exists $a \in \mathbb{N}$ such that $n = a + a$. Adding 1 to both sides of this identity, we conclude that $n+1 = (a+a)+1$. This in turn shows that $n+1$ is an odd number, which implies that $n + 1 \in X$.

Case 2. *n is odd.* Again by definition, $n = (b+b)+1$ for some $b \in \mathbb{N}$. This implies that $n+1 = ((b+b)+1)+1 = (b+1) + (b+1)$, by associativity and commutativity of addition. In particular, $n + 1$ is an even number since $b + 1 \in \mathbb{N}$ by axiom (P2). In particular, $n + 1 \in X$.

So in both cases we concluded that $n + 1 \in X$, and this establishes part (ii).

From (P5) it now follows that $\mathbb{N} \subseteq X$, and so $X = \mathbb{N}$, as we wanted to show. *Thus every natural number is even or odd.*

(b) Consider the set

$$Y := \{n \in \mathbb{N} : n \text{ is not odd, } \textbf{or } n \text{ is not even}\}.$$

Again, our goal is to show that $Y = \mathbb{N}$.

Reasoning as before, our first task is to show that $0 \in Y$. We already know that $0 \in \mathbb{N}$ from (P1). Therefore we will be done once we show that 0 is not odd. Aiming at a contradiction, assume 0 is odd. This means that there exists $a \in \mathbb{N}$ such that $0 = (a + a) + 1$, which in turn means that 0 is the successor of the natural number $a + a \in \mathbb{N}$. This clearly violates axiom (P3), and the contradiction arose from assuming that 0 is odd. Thus 0 is not odd, and so $0 \in Y$.

Additionally we want to show that $n + 1 \in Y$ if $n \in Y$. Consider an arbitrary $n \in \mathbb{N}$ such that $n \in Y$. The analysis again splits into two cases:

Case 1. *n is not odd.* It will be enough to show that $n + 1$ is not even. Assume it is, i.e., assume that there exists $b \in \mathbb{N}$ such that $n+1 = b+b$. Notice that $b \neq 0$, for $0+0 = 0$ is not the successor of any natural number by axiom (P3). In particular, b is the successor of some natural number, say b'. This means that $b' + 1 = b$, and so $n + 1 = b + b$ holds if and only if $n + 1 = (b' + 1) + (b' + 1)$ holds; moreover, $b' \in \mathbb{N}$. Since $(b' + 1) + (b' + 1) = ((b' + b') + 1) + 1$, axiom (P4) then implies that $n = (b' + b') + 1$. Since $b' \in \mathbb{N}$, this forces n to be odd, in contradiction to the hypothesis. The absurd resulting from assuming that $n + 1$ is even. Thus $n + 1$ is not even, and so $n + 1 \in Y$.

Case 2. *n is not even.* Our task is to prove that $n+1$ is not an odd number. Assume it is. Then there exists $c \in \mathbb{N}$ such that $n+1 = (c+c)+1$. This means that the natural numbers n and $c+c$ have the same successors. By axiom (P4), they must be the same. It follows that $n = c + c$. Since $c \in \mathbb{N}$, this in turn forces n to be an even number, a contradiction to the hypothesis which emerged from assuming that $n + 1$ is odd. So $n + 1$ is not odd, and so $n + 1 \in Y$.

In both cases we were able to show that $n + 1 \in Y$. Again by appealing to axiom (P5), we conclude that $\mathbb{N} \subseteq Y$. Since $Y \subseteq \mathbb{N}$ by construction, we conclude that $Y = \mathbb{N}$, as desired. Thus every natural number is *either* not odd, *or* not even. In particular, *no natural number is at the same time even and odd.*