Notes for

Harmonic and real analysis

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Recommended literature: [10, 7, 14, 13, 15]

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CHAPTER 1

Introduction

Harmonic analysis is concerned with describing, decomposing and analyzing functions and operators with some 'structure' coming from the structure of the Euclidean space. Its relevance comes from the insight that the same structures are relevant in different areas of mathematics like partial differential equations, signal processing, Fourier analysis and mathematical physics. Key notions are Fourier transform, maximal functions, square function, BMO and Hardy spaces, Calderón-Zygmund operators and their relation to partial differential equations.

1. An example

The series

$$\sum_{n=1}^{\infty} \frac{1}{n} z^n \tag{1.1}$$

converges absolute for $|z| < 1$, and uniformly on every ball of radius < 1 since

$$f(z) = \sum_{n=1}^{m} \frac{1}{n} |z|^n \le \sum_{n=0}^{m} |z|^n = \frac{1 - |z|^{m+1}}{1 - |z|} \le \frac{1}{1 - |z|}.$$

I claim that, given $\varepsilon > 0$, it converges uniformly in

$$\{z : |z| \le 1, |z - 1| > \varepsilon\}.$$

This is proven by an argument going back to Abel and mimics an integration by parts:

$$\begin{aligned} \sum_{j=n}^{m-1} \frac{z^j}{j} =& \frac{1}{m} \sum_{j=n}^{m-1} z^j + \sum_{j=n}^{m-1} \left(\sum_{k=n}^{j} z^k \right) \left(\frac{1}{j} - \frac{1}{j+1} \right) \\ =& \frac{1}{m} \sum_{j=n}^{m-1} z^j + \sum_{j=n}^{m-1} \frac{z^n - z^{j+1}}{1 - z} \left(\frac{1}{j} - \frac{1}{j+1} \right). \end{aligned}$$

To verify the formula, compare the coefficients of z^j, $n \le j \le m - 1$:

$$\frac{1}{j} = \frac{1}{m} + \sum_{l=j}^{m-1} \frac{1}{l} - \frac{1}{l+1}.$$

Then

$$\left| \sum_{j=n}^{m} \frac{z^j}{j} \right| \le \frac{4}{n\varepsilon}$$

and the partial sums are a Cauchy sequence. The limit is continuous, as a uniform limit of continuous functions. It is not hard to identify it:

$$f'(x) = \frac{1}{1-x}$$

which is the derivative of $-\ln(1-x)$ and a check at $x = 0$ shows that $f(x) = -\ln(1-x)$. Hence

$$\begin{aligned} f(z) &= -\ln_{\mathbb{C}}(1-z) = -\ln|1-z| - i\arctan\left(\frac{\operatorname{Im}(1-z)}{\operatorname{Re}(1-z)}\right) \\ &= -\ln|1-z| + i\arctan\left(\frac{\operatorname{Im} z}{1-\operatorname{Re} z}\right) \end{aligned}$$

for $|z| \le 1$, $z \neq 1$.

Define

$$g(x) := \sum_{n=1}^{\infty} \frac{(e^{ix})^n}{n} = \sum_{n=1}^{\infty} \frac{e^{inx}}{n} = -\frac{1}{2}\ln|2-2\cos x| + i\arctan\frac{\sin(x)}{1-\cos(x)}$$

since

$$|1-e^{ix}| = \sqrt{(1-\cos(x))^2 + \sin(x)^2} = \sqrt{2-2\cos(x)}.$$

By elementary geometry of the inscribed angle or the addition theorem

$$\begin{aligned} \frac{\sin(x)}{1-\cos(x)} &= \frac{2\sin(x/2)\cos(x/2)}{1-\cos^2(x/2)+\sin^2(x/2)} \\ &= \frac{\cos(x/2)}{\sin(x/2)} = \frac{\sin(\frac{\pi-x}{2})}{\cos(\frac{\pi-x}{2})} = \tan(\frac{\pi-x}{2}) \end{aligned}$$

and we obtain for the imaginary part

$$h_0(x) := \sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2}. \tag{1.2}$$

for $0 < x < 2\pi$. Define the absolute convergent series

$$h_1(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}.$$

and for $0 < x < 2\pi$

$$\begin{aligned} \int_{\pi}^{x} h_0(t)dt &= \lim_{n\to\infty} \int_{\pi}^{x} \sum_{j=1}^{n} \frac{\sin(jx)}{j} dx \\ &= -\lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{\cos(jx)}{j^2} + \sum_{j=1}^{n} \frac{\cos(j\pi)}{j^2} \right) \\ &= -h_1(x) + h_1(\pi) \end{aligned}$$

and $-h_1$ is a primitive of h_0. Thus there exists $a \in \mathbb{R}$ such that

$$h_1(x) = \frac{x^2}{4} - \frac{\pi x}{2} + a$$

and we want to determine a. Since $\int_0^{2\pi} \cos(nt)dt = \frac{1}{n}(\sin(2\pi n) + \sin(0)) = 0$ we get

$$0 = \int_0^{2\pi} h_1(t)dt = \frac{2\pi^3}{3} - \pi^3 + 2a\pi$$

and $a = \frac{\pi^2}{6}$. The evaluation at $t = 0$ gives the value of the Riemann ζ function at the value 2.

Lemma 1.1.

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{1.3}$$

Similarly

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \tag{1.4}$$

We have seen different notions of convergence.

(1) Point-wise convergence
(2) Uniform convergence
(3) Absolute convergence
(4) Convergence in L^2

in connection with Fourier series. The notions are important since we want to differentiate resp. integrate term by term.

CHAPTER 2

Fourier series

1. Definitions

We denote the one dimensional torus by $\mathbb{T} = \mathbb{R}\backslash\mathbb{Z}$. Functions on the torus are identified with periodic functions on $\mathbb{R}$ with period 1. We denote the space of Radon measures (complex Borel measures which are finite on compact sets) on a metric space X by $\mathcal{M}$ - these are objects which can be written as

$$\mu_+ + \mu_- + i\mu_{i-} - i\mu_{i-}$$

with non negative Radon measures μ_+, μ_-, μ_{i+} and μ_{i-}.

Definition 2.1. *Let $\mu \in \mathcal{T}$. We define the Fourier coefficients*

$$\hat{\mu}(n) = \hat{\mu}_n = \int_0^1 e^{-2in\pi x}\mu$$

and we write formally

$$\mu \sim \sum_{n=-\infty}^{\infty} \hat{\mu}(n) e^{2in\pi x}$$

for the relation between the series and the measure μ. If f is an integrable function we write

$$\hat{f}(n) = \hat{f}_n = \int_0^1 e^{-2in\pi x} f(x) dx$$

and

$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2in\pi x}.$$

The functions $(e^{2in\pi x})_n$ are orthonormal in the sense that

$$\widehat{e^{2i\pi nx}}(m) \int e^{2in\pi x}\overline{e^{2im\pi x}} dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

The convergence questions of the Fourier series are interesting, important, and a prototype for similar question in all areas of analysis.

We define the Dirichlet kernel through

$$\begin{aligned} S_N f(x) =& \sum_{n=-N}^{N} \hat{f}(n) e^{2in\pi x} \\ =& \sum_{n=-N}^{N} \int_{\mathbb{T}} e^{-2in\pi x'} f(x') dx' e^{2in\pi x} \\ =& \int_{\mathbb{T}} \sum_{n=-N^N} e^{2in\pi(x-x')} f(x') dx' \\ =& \int_{\mathbb{T}} D_N(x-x') f(x') dx' \\ =& D_N * f(x) \end{aligned}$$

where

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin[(2N+1)\pi x]}{\sin \pi x}.$$

We may formulate the convergence question as: When and in which sense does $D_N * f(x)$ converge to $f(x)$?

2. The convolution and Young's inequality

Motivated by the occurrence of the convolution we take a closer look at its properties. For $X = \mathbb{T}$, $\mathbb{R}$ or $\mathbb{R}^n$ we define the convolution of two continuous functions with compact support by

$$f * g(x) = \int f(x-y) g(y) dy.$$

We will need the convolution in much more general context. For that we consider

$$I_{f,g,h} = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x) g(x-y) h(y) dx dy$$

Lemma 2.2. *Suppose that* $1 \le p, q, r \le \infty$ *and*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2.$$

Then the integral defining I *is integrable for all* $f \in L^p$, $g \in L^q$ *and* $h \in L^r$ *and*

$$|I_{f,g,h}| \le \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}$$

Proof. The case when one of the exponents is ∞ and the others are 1 is simple. Hence we may assume that $1 \le p, q, r < \infty$. Let

$$\begin{aligned} F_1(x,y) =& |f(x)|^{p/r'} |g(x-y)|^{q/r'}, \\ F_2(x,y) =& |g(x-y)|^{q/p'} |h(y)|^{r/p'}, \\ F_3(x,y) =& |f(x)|^{p/q'} |h(y)|^{r/q'} \end{aligned}$$

where $'$ denotes the Hölder dual exponent

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = \frac{1}{r} + \frac{1}{r'} = 1,$$

with the obvious interpretation of the exponent ∞. Then, applying Hölder's inequality twice, since $\frac{1}{p'}+\frac{1}{q'}+\frac{1}{r'}=1$

$$\begin{aligned}
I_{fgh} \le & \int F_1F_2F_3 dxdy \\
\le & \left(\int (F_1F_2)^q dxdy\right)^{1/q} \|F_3\|_{L^{q'}} \\
\le & \|F_1\|_{L^{r'}}\|F_2\|_{L^{p'}}\|F_3\|_{L^{q'}} \\
= & \|f\|_{L^p}^{\frac{p}{r'}}\|g\|_{L^q}^{\frac{q}{r'}}\|g\|_{L^q}^{\frac{q}{p'}}\|h\|_{L^r}^{\frac{q}{p'}}\|f\|_{L^p}^{\frac{p}{q'}}\|h\|_{L^r}^{\frac{r}{q'}} \\
= & \|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r}
\end{aligned}$$

□

The lemma is the 'dual' statement to Young's inequality.

PROPOSITION 2.3 (Young's inequality). *Suppose that* $1 \le p, q, r \le \infty$ *and*

$$\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}.$$

If $f \in L^p$ *and* $g \in L^q$ *then for almost all* x *the integrand of*

$$\int f(x-y)g(y)dy$$

is integrable and it defines a function in $L^r(\mathbb{R}^n)$. *Moreover*

$$\|f*g\|_{L^r(\mathbb{R}^n)} \le \|f\|_{L^p}\|g\|_{L^q}.$$

If $r=\infty$ *then the integrand is integrable for all* x *and* $f*g$ *is a bounded continuous function.*

PROOF. Let $h \in L^{r'}$. By Lemma 2.2

$$\left|\int h(x)f(x-y)g(y)dxdy\right| \le \|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^{r'}}.$$

It is an exercise to work out the details, including the last statement. □

The convolution has nice algebraic properties.

(1) $f*g(x)=g*f(x)$
(2) $(f*g)*h(x)=f*(g*h)(x)$
(3) $f*(g+h)(x)=f*g(x)+f*h(x)$
(4) $\widehat{f*g}_n=\hat{f}_n\hat{g}_n$ for integrable functions on the torus.

The Dirichlet kernel satisfies

$$|D_N(x)| \le c\min\{|N|, \max\{\frac{1}{x},\frac{1}{1-x}\}\}$$

for $0 \le x \le 1$. If $\frac{1}{100N} \le |x| \le \frac{1}{2}$ this follows from

$$|D_N(x)| \le \frac{1}{|\sin \pi x|}$$

and for $|x| \le \frac{1}{100N}$ it follows from the Taylor expansion of $\sin N\pi x$.

We claim that there exists a constant $c>0$ so that

$$\ln(1+N)/c \le \|D_N\|_{L^1} \le c\ln(1+N).$$

The upper estimate follows by integrating the upper bound. For the lower bound we restrict the integration to

$$\{x \in [0,1] : d(Nx - \frac{1}{2}, \mathbb{Z}) \le \frac{1}{100}\}.$$

For those values x

$$|D_N(x)| \ge c \min\{N, (\sin \pi x)^{-1}\}$$

and integration gives the lower bound.

DEFINITION 2.4. *Let $0 < s < 1$, X a metric space. We say that $: X \to \mathbb{C}$ is Hölder continuous with exponent s if*

$$|f(x) - f(y)| \le c d(x,y)^s.$$

The best constant is the Hölder semi-norm.

LEMMA 2.5. *If $f : \mathbb{T} \to \mathbb{C}$ is Hölder continuous with exponent s then*

$$D_N * f \to f$$

uniformly as $N \to \infty$.

PROOF. A term-wise integration shows that $\int_{\mathbb{T}} D_N(x)dx = 1$. Thus

$$\begin{aligned}
|D_N * f(x) - f(x)| = & \left| \int_0^1 (f(x-y) - f(x)) D_N(y) dy \right| \\
\le & \int_{-\delta}^{\delta} |D_N(y)||(f(x-y) - f(x)|dy \\
& + \left| \int_{\delta}^{1-\delta} D_N(y)(f(x-y) - f(x))dy \right| \\
\le & 2c \int_{-\delta}^{\delta} |y|^{-1}|y|^s dy \|f\|_{\dot{C}^s} \\
& + \left| \int_{\delta}^{1-\delta} \frac{f(x-y) - f(x)}{\sin(\pi y)} \sin((2N+1)\pi y) dy \right| \\
\le & \frac{4c}{s}\delta^s \|f\|_{\dot{C}^s} + \left| \int_{\delta}^{1-\delta} h_x(y) \sin((2N+1)\pi y) dy \right|
\end{aligned}$$

It remains to estimate the second term on the right hand side. We denote it by B and use that $\sin((2N+1)\pi y) = -\sin((2N+1)\pi(y + \frac{1}{2N+1}))$. Thus

$$\begin{aligned}
B \le & \left| \int_{\delta}^{1-\delta} (h_x(y) - h_x(y - \frac{1}{2N+1}) \sin((2N+1)\pi y) dy \right| \\
& + \left| \int_{\delta - \frac{1}{2N+1}} |h_x(y)| dy \right| + \left| \int_{1-\delta}^{1-\delta+\frac{1}{2N+1}} h_x(y) dy \right| \\
\le & (2N+1)^{-s}\delta^{-1}\|f\|_{\dot{C}^s} + (2N+1)^{-1}\delta - 2\|f\|_{sup}.
\end{aligned}$$

The estimate holds for all δ. We choose $\delta = N^{-\frac{s}{3}}$ and obtain

$$|D_N f(x) - f(x)| \le c \left(N^{-\frac{s^2}{3}} + N^{-\frac{2s}{3}} + N^{-1+\frac{2s}{3}} \right).$$

□

We can improve the convergence by using Cesáro means resp. the Fejér kernel.

$$\sigma_N f = \frac{1}{N}\sum_{n=1}^{N-1} D_n * f(x)$$
$$K_N * f(x)$$

where

$$K_N = \sum_{n=0}^{N-1} D_n(x) = \frac{1}{N}\left(\frac{\sin N\pi x}{\sin \pi x}\right)^2.$$

It satisfies

$$\hat{K}_N(n) = \left(1 - \frac{|n|}{N}\right)_+$$

which is seen by checking the Fourier transform

$$0 \le K_N(x) \le \frac{C}{N}\min\{N^2, |\sin(\pi x)|^{-2}\} \tag{2.1}$$

by definition (left hand side) and as for D_N (right hand side),

$$\int_0^1 K_n(x) = \int_0^1 |K_N(x)| = 1$$

which we can read off the Fourier coefficients and

$$|K_N'(x)| \le C\min\{N^2, |\sin(\pi x)|^{-2}\}. \tag{2.2}$$

which is a straight forward calculation for $\frac{1}{100N} \le x \le 1 - \frac{1}{100N}$, and which follows from Taylor expansion in the remaining interval.

We call a function *approximate identity.*

Definition 2.6. *The family Φ_n of functions on $\mathbb{T}$, $\mathbb{R}$ or $\mathbb{R}^n$ is called approximate identity if*

1. $\int_0^1 \Phi_n(x)dx = 1$
2. $\sup_N \int_0^1 |\Phi_n(x)|dx < \infty$
3. *For all $\delta > 0$ one has $\int_\delta^{1-\delta} |\Phi_n(x)|dx \to 0$ as $n \to \infty$*

Proposition 2.7. *Let Φ_n be an approximate identity. If $f \in C(\mathbb{T})$ then*

$$\Phi_n * f \to f$$

uniformly as $n \to \infty$. If $f \in L^p(\mathbb{T})$, $1 \le p < \infty$, then

$$\|\Phi_n * f - f\|_{L^p(\mathbb{T})} \to 0$$

as $n \to \infty$. If $\mu \in \mathcal{M}(\mathbb{T})$ then

$$\Phi_n * \mu \to \mu$$

in the sense of measures.

PROOF. The proof uses some constructions which we will need often. Let $f : \mathbb{T} \to \mathbb{R}$ be continuous, and Φ_n and approximate identity. Then

$$\begin{aligned} f(x) - f * \Phi_n(x) &= \int_0^1 (f(x) - f(x-y))\Phi_n(y)dy \\ &= \int_{-\delta}^{\delta} (f(x) - f(x-y))\Phi_n(y)dy \\ &\quad + \int_{\delta}^{1-\delta} (f(x) - f(x-y))\Phi_n(y)dy \end{aligned}$$

We define the modulus of continuity

$$\omega_f(t) = \sup_{|x-y|<t} |f(x) - f(y))|.$$

Then $\lim_{t\to 0} \omega_f(t) = 0$ since f is uniformly continuous. Then

$$|f(x) - f * \Phi_n(x)| \le 2\omega(\delta)\|\Phi_n\|_{L^1} + 2\|f\|_{sup} \int_{-\delta}^{1-\delta} |\Phi_n(y)|dy$$

and

$$\limsup_{n\to\infty} \sup_x |f(x) - f * \Phi_n(x)| \le 2\omega(\delta) \sup_n \|\Phi_n\|_{L^1}$$

This holds for all δ, hence the lim sup is zero.

Assume now that $f \in L^p$ for some $1 \le p < \infty$. Continuous functions are dense in $L^p(\mathbb{T})$. Given ε there exists a continuous function f_ε such that

$$\|f - f_\varepsilon\|_{L^p(\mathbb{T})} \le \varepsilon.$$

Then

$$I = \|f - f * \Phi_n\|_{L^p} \le \|f_\varepsilon - f_\varepsilon * \Phi_n\|_{L^p} + \|f - f_\varepsilon\|_{L^p(\mathbb{T})} + \|(f - f_\varepsilon) * \Phi_n\|_{L^p(\mathbb{T})}.$$

By Young's inequality with $p = r$ and $q = 1$

$$\|(f - f_\varepsilon) * \Phi_n\|_{L^p(\mathbb{T})} \le \|\Phi_n\|_{L^1}\|f - f_\varepsilon\|_{L^p}$$

Then

$$I_n \le \|f_\varepsilon - f_\varepsilon * \Phi_n\|_{sup} + \varepsilon(1 + \|\Phi_n\|_{L^1})$$

and

$$\begin{aligned} \limsup_{n\to\infty} I_n &\le \limsup_{n\to\infty} \|f_\varepsilon - f_\varepsilon * \Phi_n\|_{sup} + \varepsilon(1 + \|\Phi_n\|_{L^1}) \\ &\le \varepsilon(1 + \|\Phi_n\|_{L^1}) \end{aligned}$$

This holds for all ε and hence the lim sup is 0.

The convolution of a measure with an L^1 function is a integrable function. The extension of the dual Young's inequality is easy.

Let μ be a measure and h a continuous function. If (Φ_n) is an approximate identity then the same is true for $\tilde{\Phi}_n(t) = \Phi_n(-t)$ and

$$\int \mu * \Phi_n h(x)dx = \int \tilde{\Phi}_n * h(x)\mu \to \int h\mu$$

This is the definition of the convergence of measures. □

COROLLARY 2.8. *The trigonometric polynomials are dense in L^p for $1 \le p < \infty$ and in $C(\mathbb{T})$. For $f \in L^2$ the identity of Plancherel*

$$\|f\|_{L^2}^2 = \sum_{n=-\infty}^{\infty} |\hat{f}_n|^2 \tag{2.3}$$

holds. The complex exponentials $(e^{2\pi inx})_n$ are an orthonormal basis of $L^2(\mathbb{T})$ and for $f, g \in L^2$ one has Parsevals identity

$$\int f\bar{g}dx = \sum \hat{f}_n \overline{\hat{g}_n}.$$

The inner product of $L^2(\mathbb{T})$ is given by the formula in the corollary. The space is a Hilbert space i.e. $\|f\|_{L^2}^2 = \langle f, f\rangle$ is a norm, and the space if complete with this norm.

PROOF. (K_n) is an approximate identity. By Proposition 2.7 we have $f * K_n \to f$ in L^p for any $f \in L^p$ if $p < \infty$. Now $\hat{K}_n$ is a trigonometric polynomial, and hence $f * K_n$ is a trigonometric polynomial. This implies the density. The identity of Plancherel is trivial for trigonometric polynomials: When we expand them and integrate all nondiagonal terms will give 0. Now

$$\|f\|_{L^2}^2 = \lim_{n\to\infty} \|f * K_n\|_{L^2}^2 = \lim_{n\to\infty} \sum_{j=-n}^{n} |\hat{f}_j|^2 (1 - \frac{j}{n})^2 = \sum_{j=-\infty}^{\infty} |\hat{f}_j|^2$$

by monotone convergence. Let $f, g \in L^2(\mathbb{T})$. Then, using Plancherels formula

$$\begin{aligned}\frac{1}{2}\int_0^1 f\bar{g} + g\bar{f}dt =& \frac{1}{4}\int_0^1 |f+g|^2 - |f-g|^2 dx \\ =& \frac{1}{4}\sum_{j=-\infty}^{\infty} |\hat{f}_n + \hat{g}_n|^2 - |\hat{f}_n - \hat{g}_n|^2 \\ =& \frac{1}{2}\sum_{j=-\infty}^{\infty} \hat{f}_n\overline{\hat{g}_n} + \hat{g}_n\overline{\hat{f}_n}\end{aligned}$$

which is Parsevals identity for the real part for the formula. We replace f by if to obtain imaginary part of the formula. The complex exponentials $e^{2\pi inx}$ are clearly orthonormal. if $\int fe^{2\pi inx}dx = n$ for all n then $f = 0$ by Plancherels identity. □

3. L^p convergence of partial sums

PROPOSITION 2.9. *Let $1 \le p < \infty$. Then the following is equivalent:*

1. *For all $f \in L^p$ we have $\|D_N * f - f\|_{L^p} \to 0$*
2. $\sup_n \frac{\|D_N * f\|_{L^p}}{\|f\|_{L^p}} < \infty$.

PROOF. (1) $\Longrightarrow$ (2) is a consequence of the uniform boundedness principle. The reverse implication follows from the density of trigonometric polynomials and Young's inequality. □

COROLLARY 2.10. *There exists $f \in L^1$ so that $D_N * f$ does not converge to f in L^1. There exists $f \in C(\mathbb{T})$ so that $D_N * f$ does not converge uniformly to f.*

PROOF. We have seen that

$$\|D_N\|_{L^1} \geq c\ln(1+N).$$

Let $f_\delta(x) = (2\delta)^{-1}\chi_{|x|\leq\delta}$. Then $\|f\|_{L^1} = 1$ and, if $\delta < \frac{1}{4N}$, $\|D_N * f\|_{L^1} \geq c\ln(1+N)$. This contradicts (2) and hence there exists $f \in L^1$ so that $D_N * f$ does not converge to f in L^1. If $f(y) = \frac{D_N(-y)}{|D_N(-y)|}$ if $D_N(y) \neq 0$ and 0 otherwise then

$$|D_N * f(0)| \geq c\ln(1+N).$$

By dominated convergence

$$\lim_{\varepsilon\to 0} D_N * \frac{D_N}{\varepsilon + |D_N|}(0) \to D_N * f(0)$$

and hence

$$\sup_{N\geq 1} \sup_{f\in C(\mathbb{T}), |f|\leq 1} \|D_N * f\|_{sup} = \infty.$$

Arguing as in the proposition with the uniform boundedness principle this implies that there is $f \in C(\mathbb{T})$ such that $D_N * f$ does not converge uniformly to f. □

The variant of the argument gives a stronger statement: There exist $f \in L^1$ (resp. $f \in C(\mathbb{T})$ so that $\limsup_{N\to\infty} \|D_N * f\|_{L^1} \to \infty$ (resp. $\limsup_{N\to\infty} \|D_N * f\|_{sup} = \infty$). Exercise: Find such functions.

4. Regularity and Fourier series

PROPOSITION 2.11 (Bernstein's inequality). *Let f be a trigonometric polynomial with $\hat{f}(k) = 0$ for $|k| > n$. Then*

$$\|f'\|_{L^p} \leq Cn\|f\|_{L^p}$$

PROOF. We define the de la Vallée Poussin kernel

$$\begin{aligned} V_N(x) =& \frac{1}{N}\sum_{k=N+1}^{2N}\sum_{j=1-k}^{k-1} e^{2\pi i j x} \\ =& 2K_{2N}(x) - K_N(x) \\ =& \frac{1}{N}\left(\frac{\sin^2 2N\pi x - \sin^2 N\pi x}{\sin^2 \pi x}\right) \end{aligned}$$

Then, by Young's inequality (and a tedious bound of the L^1 norm)

$$\begin{aligned} \|f'\|_{L^p} =& \|(V_N * f)'\|_{L^p} \\ =& \|V_N' * f\|_{L^p} \\ \leq& \|V_N'\|_{L^1}\|f\|_{L^p} \\ \leq& CN\|f\|_{L^p}. \end{aligned}$$

□

We define Sobolev spaces on the torus.

DEFINITION 2.12. *Let $s \in \mathbb{R}$. We define the Hilbert space $H^s(\mathbb{T})$ as the completion of the trigonometric polynomials in the norm*

$$\|f\|_{H^s} : \left(\sum_{j=-\infty}^{\infty} (1+n^2)^s |\hat{f}(n)|^2 \right)^{1/2} .$$

If $s \ge 0$ then $H^s(\mathbb{T}) \subset H^0(\mathbb{T}) = L^2(\mathbb{T})$. If $f \in H^1$ then $f' \in L^2$ and

$$\|f'\|_{L^2}^2 + \|f\|_{L^2}^2 = \|f\|_{H^1}^2.$$

This holds for trigonometric polynomials, and we use it to define the derivative of functions in H^1.

THEOREM 2.13. *Let $0 < s \le 1$. Then $C^s \subset H^\sigma$ for all $\sigma < s$.*

PROOF. We claim that

$$\sum_{2^j \le |n| \le 2^{j+1}} |f(n)|^2 \le c 2^{-2js} \|f\|_{C^s(\mathbb{T})}^2. \tag{2.4}$$

Then

$$\sum_{n=-\infty}^{\infty} |n|^{2\sigma} |\hat{f}(n)|^2 \le 2 \sum_{j=0}^{\infty} 2^{2j\sigma} \sum_{2^j \le |n| \le 2^{j+1}} |f(n)|^2 \le C \sum_{j=0}^{\infty} 2^{2j(\sigma-s)} \|f\|_{C^s}$$

To prove the claim (2.4) we observe that it follows from

$$\begin{aligned} \|K_n * f - f\|_{L^2} \le & \sup_x |K_n * f(x) - f(x)| \\ \le & 2 \int_0^{1/2} |K_n(y)||y|^s dy \|f\|_{C^s} \\ \le & c n^{-s} \|f\|_{C^s} \end{aligned}$$

(see (2.1)) with $n = 2^{j-1}$ since

$$\sum_{2^j \le |n| \le 2^{j+1}} |f(n)|^2 \le \|K_{2^{j-1}} * f - f\|_{L^2}.$$

□

5. Complex Interpolation

Holomorphic functions satisfy a maximum principle.

LEMMA 2.14. *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ holomorphic. If $z_0 \in U$ and*

$$|f(z_0)| = \max\{|f(z)| : z \in U\}$$

then f is constant.

PROOF. 1) There is nothing to show if $f(z_0) = 0$. Otherwise we divide by $f(z_0)$ and may assume that $f(z_0) = 1$ The Taylor series

$$f(z) = 1 + \sum_{j=1}^{\infty} a_j (z - z_0)^j$$

converges in a neighborhood of z_0. There is a first non-vanishing coefficient a_m with $m \geq 1$. There exists b with $b^m = a_m$. We consider $g(z) = f(z/b)$. It has the form

$$g(z) = 1 + z^m + \sum_{n=m+1}^{\infty} a_n z^n.$$

Then

$$\frac{d}{dt}|g(1+t)|\Big|_{t=0} = m$$

and we obtain a contradiction.

2) By the Cauchy integral formula, resp. the mean value property of harmonic functions, for $r > 0$ and small

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt$$

This implies that the maximum is assumed at the boundary. If it is assumed in an interior point then $|f(z)|$ is constant and the holomorphic function $z \to \ln_{\mathbb{C}} f(z)$ has constant real part. The Cauchy-Riemann equations imply now that f is constant. □

Lemma 2.15 (Three lines theorem). *Let $f : \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\} \to \mathbb{C}$ be bounded, continuous and holomorphic in the interior. Then*

$$|f(x+iy)| \leq \left(\sup_y |f(y)|\right)^{1-x} \left(\sup_y |f(1+iy)|\right)^x.$$

Proof. Let

$$f_\varepsilon(z) = e^{\varepsilon z^2} f(z) \left(\sup_y |f(iy)|\right)^{z-1} \left(\sup_y |f(1+iy)|\right)^{-z}.$$

Then

$$|f_\varepsilon| \leq ce^{\varepsilon(1-y^2)} \to 0 \qquad \text{as } y \to 0.$$

Thus there exists $z = x + iy$ with $0 \leq x \leq 1$ where $|f_\varepsilon|$ is maximal. Then either f_ε is constant and hence identically 0, or the maximum is assumed at the boundary. But then

$$\sup |f_\varepsilon(z)| \leq \max\{\sup_y |f_\varepsilon(iy)|, \sup_y |f_\varepsilon(1+iy)|\}$$

and hence

$$|f(z)| \leq e^\varepsilon \left(\sup_y |f(y)|\right)^{1-x} \left(\sup_y |f(1+iy)|\right)^x$$

□

Theorem 2.16 (Riesz-Thorin interpolation theorem). *Let μ and ν be measures on X resp. Y, T_z maps characteristic functions of measurable sets of bounded μ measure to functions which are integrable over sets of finite measure, for $0 \leq \operatorname{Re} z \leq 1$ such that for all measurable sets $A \subset X$ and $B \subset Y$ of finite measure the map*

$$z \to \int (T_z \chi_A(y)) \chi_B(y) dy$$

is bounded and continuous and holomorphic in the interior of the strip. Suppose that

$$1 \le p_0, p_1, q_0, q_1 \le \infty$$

and,

$$\frac{1}{p_z} = \frac{\operatorname{Re} z}{p_1} + \frac{1-\operatorname{Re} z}{p_0}$$
$$\frac{1}{q_z} = \frac{\operatorname{Re} z}{q_1} + \frac{1-\operatorname{Re} z}{q_0}.$$

Suppose that

$$\left| \int (T_{iy'} f) g \nu \right| \le C_0 \|f\|_{L^{p_0}} \|g\|_{L^{q_0'}}$$
$$\left| \int (T_{1+iy'} f) g \nu \right| \le C_1 \|f\|_{L^{p_1}} \|g\|_{L^{q_1'}}.$$

Then T_λ has a unique extension to a continuous linear operator

$$T_\lambda : L^{p_z}(\mu) \to L^{q_z}(\nu)$$

with norm $C_0^{1-\operatorname{Re} z} C_1^{\operatorname{Re} z}$.

Proof. By duality and density it suffices to prove

$$\left| \int (T_z f) g \nu \right| \le C_0^{1-\operatorname{Re} z} C_1^{\operatorname{Re} z} \|f\|_{L^p} \|g\|_{L^{q'}}$$

for $0 < t < 1$, for finite sums of characteristic functions of sets of finite measure. We fix z_0, $p = p_{z_0}$, $q = q_{z_0}$, f and g a finite sum of characteristic functions of sets of finite measure on X resp Y,

$$f_z(x) = |f(x)|^{\frac{t-z}{t}\frac{p-p_0}{p_0}} f(x), \quad g_\lambda(y) = |g(x)|^{\frac{t-z}{t}\frac{q'-q_0'}{q_0'}} g(y)$$
$$f_t(x) = f(x) \qquad g_t(y) = g(y)$$
$$|f_{iy'}(x)|^{p_0} = |f(x)|^p, |g_{iy'}(y)|^{q_0'} = |g(y)|^{q'}$$
$$(\operatorname{Re} z_0 - 1)\left(\frac{p}{p_0} - 1\right) = \operatorname{Re} z_0 \left(\frac{p}{p_1} - 1\right)$$
$$|f_{1+iy'}(x)|^{p_1} = |f(x)|^p, |g_{1+iy'}(y)|^{q_1'} = |g(y)|^{q'}.$$

We taking the difference quotients and using dominated convergence

$$\frac{d}{dz} \int (T_z f_z)(y) g_z(y) \nu(y) = \int (\frac{d}{dz} T_z) f_z(y) g_z(y) \nu(y) + \int (T_\lambda \frac{d}{dz} f_z) g_z \nu(y)$$
$$+ \int (T_z f_z) \frac{d}{dz} g_z \nu(y)$$

and hence $\int T_z f_z g_z \nu$ is holomorphic in z. Boundedness and continuity are immediate.

By Hölder's inequality and the construction

$$\begin{aligned} \left| \int (T_{iy'} f_{iy'}) g_{iy'} \nu(y) \right| \le & \|T_{iy'} f_{iy'}\|_{L^{q_0}} \|g_{iy'}\|_{L^{q_0'}} \\ \le & c_0 \|f_{iy'}\|_{L^{p_0}} \|g_{iy'}\|_{L^{q_0'}} \\ \le & c_0 \|f\|_{L^p}^{p/p_0} \|g\|_{L^{q'}}^{q'/q_0'}. \end{aligned}$$

and

$$\left|\int (T_{1+iy'}f_{1+iy'})g_{1+iy'}dy\right| \le C_1\|f\|_{L^p}^{p/p_1}\|g\|_{L^{q'}}^{q'/q_1'}.$$

By the three lines theorem

$$\begin{aligned}\left|\int T_{z_0}fg\right| \le& \Big(C_0\|f\|_{L^p}^{p/p_0}\|g\|_{L^{q_0'}}^{q/q_0'}\Big)^{1-\operatorname{Re} z_0}\Big(C_1\|f\|_{L^p}^{p/p_1}\|g\|_{L^{q'}}^{q'/q_1'}\Big)^{\operatorname{Re} z_0}\\ =& C_0^{1-\operatorname{Re} z_0}C_1^{\operatorname{Re} z_0}\|f\|_{L^p}^{p\left(\frac{1-\operatorname{Re} z_0}{p_0}+\frac{\operatorname{Re} z_0}{p_1}\right)}\|g\|_{L^{q'}}^{q'\left(\frac{(1-\operatorname{Re} z_0)}{q_0'}+\frac{\operatorname{Re} z_0}{q_1'}\right)}\\ =& C_0^{1-\operatorname{Re} z_0}C_1^{\operatorname{Re} z_0}\|f\|_{L^p}\|g\|_{L^{q'}}.\end{aligned}$$

□

The lemma of Schur is a consequence.

LEMMA 2.17 (Schur). *Let $K : X \times Y \to \mathbb{C}$ be $\mu \times \nu$ measurable and suppose that*

$$\sup_x \int |K(x,y)|\nu(y) \le C_1, \qquad \sup_y \int |K(x,y)|\mu(x) \le C_\infty$$

Let $1 \le p \le \infty$ The linear map defined on characteristic functions by

$$T(\chi_A)(y) = \int_A K(x,y)d\mu(x)$$

has a unique extension to a continuous linear map $T : L^p(X,\mu) \to L^p(Y,\nu)$ which satisfies

$$\|Tf\|_{L^p(Y,\nu)} \le C_1^{\frac{1}{p}}C_\infty^{1-\frac{1}{p}}\|f\|_{L^p(X,\mu)}.$$

PROOF. If $f \in L^\infty(X,\mu)$ then $|Tf(y)| \le C_\infty\|f\|_{L^\infty}$. If $f \in L^1$ then

$$\begin{aligned}\int |\int Tf(y)|\nu(y) \le& \int |K(x,y)||f(x)|\mu(x)\nu(y)\\ \le& \sup_x \int |K(x,y)|d\nu(y)\|f\|_{L^1(X,\mu)}\\ \le& C_1\|f\|_{L^1}.\end{aligned}$$

This holds first for simple functions. There is a unique extension to $T : L^1(X,\mu) \to L^1(Y,\nu)$ bounded by C_1 and a unique extension to $L^\infty(X,\mu) \to L^\infty(Y,\nu)$ bounded by C_∞. The Riesz-Thorin theorem implies the full statement. □

Alternatively we may estimate as in Young's inequality:

$$\begin{aligned}&\left|\int f(x)K(x,y)g(y)d\mu(x)d\nu(y)\right|\\ &\le\|f(x)|K(x,y)|^{1/p}\|_{L^p(\mu\times\nu)}\||K(x-y)|^{1/p'}g(y)\|_{L^{p'}(\mu\times\nu)}\\ &\le\|f\|_{L^p(\mu)}\|g\|_{L^{p'}(\nu)}\sup_x\||K(x,.)|^{1/p}\|_{L^p(\nu)}\sup_<\||K(.,y()|^{1/p'}\|_{L^{p'}(\mu)}.\end{aligned}$$

This second proof gives a stronger statement: Existence of the defining integral for almost all y.

We could derive the bound in Young's inequality via the Theorem of Riesz-Thorin from the extremal cases $p = 1$, $q = \infty$ and $p = 1$, $q = 1$. With this type of argument we would however loose the existence of the integral for almost all y. Up to that Young's inequality with $p = 1$ is a special case of Schur's lemma.

LEMMA 2.18 (The Hausdorff-Young inequality). *Let $f \in L^p(\mathbb{T}, \mathbb{C})$, $1 \le p \le 2$. Then*

$$\|\hat{f}\|_{l^{p'}} \le \|f\|_{L^p}$$

PROOF. The case $p = 2$ is the Plancherel identity. The case $p = 1$ is trivial. If

$$\frac{1}{p} = \lambda + \frac{1-\lambda}{2}$$

then

$$\frac{1}{p'} = \frac{1-\lambda}{2}$$

The assertion follows from Theorem 2.16 with μ the Lebesgue measure and ν the counting measure. □

6. The Hardy-Littlewood maximal function and real interpolation

DEFINITION 2.19. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define the (uncentered) Hardy-Littlewood maximal function as*

$$Mf(x) = \sup_{x \ni B_R(y)} |B_R(y)|^{-1} \int_{B_R(y)} |f(z)| dz$$

THEOREM 2.20 (Hardy-Littlewood maximal function). *Let $1 \le p \le \infty$ and $f \in L^p(\mathbb{R}^n)$. The Hardy-Littlewood maximal function Mf is measurable and finite almost everywhere. It satisfies*

$$\|Mf\|_{L^p(\mathbb{R}^n)} \le c(n) \frac{p}{p-1} \|f\|_{L^p(\mathbb{R}^n)}.$$

and

$$m^n(\{x : Mf(x) > \lambda\}) \le \frac{3^n}{\lambda} \|Mf\|_{L^1}$$

DEFINITION 2.21. *Let $f \in L^1_{loc}$. We call $x \in \mathbb{R}^n$ Lebesgue point if*

$$\lim_{R \to 0} \sup_{B_R(x_1) \cap B_r(x_2) \ni x, r \le R} \left| (B_R(x_1))^{-1} \int f(y) dy - (B_r(x_2))^{-1} \int f(y) dy \right| = 0$$

We may (and usually do) assume that at a Lebesgue point f is equal to the limit of the averages as $R \to 0$.

THEOREM 2.22. *Let $f \in L^1_{loc}$. Then there is a set A of measure 0 so that all point in $\mathbb{R}^n \backslash A$ are Lebesgue points.*

PROOF. We may assume that f is compactly supported and integrable. Given $\varepsilon > 0$ there exists a continuous compactly supported function g with

$\|f-g\|_{L^1} < \varepsilon$. Since for continuous functions all points are Lebesgue points it remains to consider $h = f - g$. Then

$$\begin{aligned} A_t =& \{x : \limsup_{r,R} |f_{B_R(x_1)} - f_{B_r(x_2)}| > t\} \\ &\subset \{x : \limsup_{r,R} |h_{B_R(x_1)} - h_{B_r(x_2)}| > t\} \\ &\subset \{x : Mh(x) > t/2\} \end{aligned}$$

and its measure is bounded by $\frac{c\varepsilon}{t}$. We let $\varepsilon \to 0$ to see that

$$m^n(A_t) = 0.$$

Denote the set of Lebesgue points by L. Then

$$\mathbb{R}^n \backslash L = \bigcup_j A_{1/j}$$

is a countable union of sets of measure 0, hence its union has measure 0. □

Proof of Theorem 2.20. The proof consists of two steps, the first being the proof of the weak type inequality for $p = 1$, and the second being an interpolation argument.

Lemma 2.23 (Covering argument of Vitali). *Let (X, d) be a metric space and $(B_i)_{1\le i\le N} = (B_{r_i}(x_i))_{1\le i\le N}$ by a finite set of balls. Then there exists a pairwise disjoint subset $(B_{i_j})_{1\le j\le M}$ so that*

$$\bigcup_{i=1}^N B_i \subset \bigcup_{j=1}^M B_{3r_{i_j}}(x_{i_j}).$$

Proof. We choose the balls recursively. Suppose we have chosen disjoint balls B_{i_j} for $j < m$. Let $B_{i_{jm}}$ be the ball of the largest radius which is disjoint to the previous ones. If there is no such ball we are done. This process ends at some point. We have to verify the covering statement. Let $B_{r_i}(x_i)$ be one of the balls. If it has been chosen it is certainly contained in the union to the right. If not there is a largest index m so that $r_{i_m} \ge r_i$. Since we did not choose $B_{r_i}(x_i)$ in this step it has a nonempty intersection with one of the balls $B_{r_{i_j}}$ with $j \le m$. But then $r_i \le r_{i_j}$ and $B_{r_i}(x_i) \subset B_{3r_{i_j}}(x_{i_j})$. □

Let $f \in L^1(\mathbb{R}^n)$ and $t > 0$. The set $U = \{x : Mf > t\}$ is open. For each point $x \in U$ there is a ball $B_r(y) \ni x$ with

$$\int_{B_r(y)} |f(y)|dy > \lambda m^n(B_r(y)).$$

Let $K \subset U$ be a compact set. Since K is compact, and covered by these balls there exists a finite number of such balls $(B_{r_i}(x_i))_{1\le i\le N}$ which cover K. By Lemma 2.23 there is a subset of disjoint balls $(B_{r_{i_j}}(x_{i_j}))$

$$K \subset \bigcup_{i=1}^N B_{r_i}(x_i) \subset \bigcup_{j=1}^M B_{3r_{i_j}}(x_{i_j})$$

Thus

$$\begin{aligned} m^n(K) \le & m^n(K)\left(\bigcup_{j=1}^M B_{3r_{i_j}}(x_{i_j})\right) \\ \le & 3^n m^n\left(\bigcup_{j=1}^M B_{r_{i_j}}(x_{i_j})\right) \\ \le & 3^n t^{-1}\int_{Mf>t} |f| \\ \le & \frac{3^n}{t}\|f\|_{L^1} \end{aligned}$$

and

$$m^n(\{x : Mf(x) > t\}) = \sup_K m^n(K) \le \frac{3^n}{t}\|f\|_{L^1}.$$

Clearly $Mf(t) \le \|f\|_{L^\infty}$. Let $1 \le p < \infty$. There is the bath tube representation

$$\int |f| dx = m^{n+1}(\{(x,t) : 0 \le t < |f(x)|\}) = \int_0^\infty m^n(\{|f| > t\}) dt \tag{2.5}$$

and

$$\int |f|^p dx = \int_0^\infty m^n(\{|f| > s^{1/p}\}) ds = p\int_0^\infty t^{p-1} m^n(\{|f| > t\}) dt. \tag{2.6}$$

Given $t > 0$ we define

$$f_t = \begin{cases} f & \text{if } |f| \le t \\ t\frac{f}{|f|} & \text{if } |f| > t \end{cases}$$

and

$$f^t = f - f_t.$$

The maximal function is sublinear, i.e.

$$Mf(x) \le Mf_t(x) + Mf^t(x).$$

Thus

$$\begin{aligned} \|Mf\|_{L^p}^p = & 2^p p\int_0^\infty t^{p-1} m^n(\{Mf > 2t\}) dt \\ \le & 2^p p\int_0^\infty t^{p-1}\Big(m^n(\{Mf_t > t\}) + m^n(\{Mf^t > t\})\Big) dt \\ \le & 2^p p\int_0^\infty t^{p-1} m^n(\{Mf^t > t\}) dt \\ \le & 2p3^n\int_0^\infty t^{p-2}\int_{|f|>t} |f| dx dt \\ = & 2^p p 3^n\int_{\mathbb{R}^n} |f(x)|\int_0^{|f(x)|} t^{p-2} dt \\ = & \frac{2^p 3^n}{p-1}\|f\|_{L^p}^p \end{aligned}$$

and

$$\|Mf\|_{L^p} \le \left(\frac{23^n}{p-1}\right)^{1/p} \|f\|_{L^p}.$$

□

By Tschebychevs inequality

$$\mu(\{|f| > \lambda\}) \le \lambda^{-p} \|f\|_{L^p}^p.$$

We define the weak norm by

$$\|f\|_{L^p_w(\mu)} = \sup_\lambda \lambda \mu(\{|f| > \lambda\})^{1/p} \tag{2.7}$$

This is an abuse of notation, since the 'norm' does in general not satisfy the triangle inequality but only

$$\|f+g\|_{L^p_w} \le C\left(\|f\|_{L^p_w} + \|g\|_{L^p_w}\right). \tag{2.8}$$

If $p = \infty$ we set $L^\infty_w = L^\infty$.

THEOREM 2.24 (Marcinkiewicz). *Let $1 \le p_0 < p_1 \le \infty$, $1 \le q_0 \ne q_1 \le \infty$. Suppose that T is a sublinear operator mapping the span of characteristic function of μ measurable sets of finite to functions which are ν integrable over ν measurable sets of finite ν measure, such that,*

(1) *For all finite collections of measurable sets $A_j \subset X$ and $B \subset Y$ of finite measure the map*

$$\mathbb{R}^N \ni (a) \to \int_B T \sum a_j \chi_{A_j} \nu$$

is continuous.

(2) *$T(tf)(y) = tT(f)(y)$ for $t > 0$ and $|T(f+g)(x)| \le |Tf(x)| + |Tg(x)|$ almost everywhere.*

(3) $\left|\int T(\chi_A)\chi_B dy\right| \le \min\{c_0 \mu(A)^{1/p_0} \mu^{1/q_0'}(B), c_1 \mu(A)^{1/p_1} \mu^{1/q_1'}(B)\}$

If $0 < \lambda < 1$

$$\frac{1}{p} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}, \frac{1}{q} = \frac{1-\lambda}{q_0} + \frac{\lambda}{q_1}$$

then T defines a unique continuous sublinear operator from L^p_w to L^q_w.

$$\|Tf\|_{L^q_w} \le c(p,q,\lambda) \|f\|_{L^p_w}$$

and

$$\left|\int_Y T(t\chi_A) g(y) d\nu\right| \le c(p,q,\lambda) |t| \mu(A)^{1/p} \|g\|_{L^{q'}_w}$$

If $p \le q$ then $T : L^q \to L^p$ and

$$\|Tf\|_{L^q} \le c(p,q,\lambda) \|f\|_{L^p}.$$

We apply the Marcinkiewicz interpolation theorem to an important version of Young's inequality before we prove it.

LEMMA 2.25 (Weak Young's inequality). *Let*

$$1 < p, q, r < \infty, \qquad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$$

and $f \in L^p$ and $g \in L^p_w$. Then the integral

$$f * g(x) = \int f(x-y)g(y)dy$$

exists for almost all x and

$$\|f * g\|_{L^r} \le c\|f\|_{L^p}\|g\|_{L^q_w}.$$

PROOF. Truncating $g = g_1 + g^1$ as above it is not hard to see that the integral exists for almost all x. By Young's inequality

$$\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}$$

for all such triple. We fix p and $f \in L^p$ and define

$$Tg = f * g, \qquad T : L^q \to L^r$$

for all admissible triple. Thus, by duality,

$$\left|\int Tghdx\right| \le \|g\|_{L^q}\|h\|_{L^{r'}}$$

and we can apply the interpolation theorem of Markinciewicz for all admissible exponent beside $q = 1$ or $r = \infty$:

$$\|f * g\|_{L^r_w} \le c\|f\|_{L^p}\|g\|_{L^q_w}.$$

Now we fix $g \in L^q_w$ and define

$$Tf = f * g, T : L^p_w \to L^r_w.$$

This implies

$$\left|\int_B T\tilde{f}\chi_A dx\right| \|\tilde{f}\|_{L^\infty}|A|^{1/p}|B|^{1/r'},$$

and, since $p \le r$,

$$\|Tf\|_{L^r} \le c\|f\|_{L^p}\|g\|_{L^q_w}$$

for triples which we obtain by interpolation, which are those of the lemma. □

The assumption of the theorem implies for all simple functions

$$\left|\int_B Tf\chi_A g\nu\right| \le c\|f\|_{L^\infty(\mu)}\|g\|_{L^\infty(\nu)}\mu(A)^{1/p_0}\nu(B)^{1/q_0'} \tag{2.9}$$

with a similar estimate for the index 1. Allowing for a factor 4 in the constant it suffices to verify this for nonnegative functions f and g. We can then write

$$f = \sum_j f_j\chi_{A_j}, g = \sum g_j\chi_{B_j}$$

with $0 \le f_j, g_j$, $\sum f_j = \|f\|_{L^\infty}$, $\sum g_j = |g\|_{L^\infty}$, $A_j \subset A_{j_1}$ and $B_j \subset B_{j-1}$ for all j. We expand the sum and use the sublinearity to arrive at (2.9).

LEMMA 2.26. *If $1 < p < \infty$ and $f \in L^p_w$ the inequalities*

$$\frac{1}{2}\|f\|_{L^p_w} \le \sup_{\mu(A)>0} \mu(A)^{-1/p'}\left|\int_A f\mu\right| \le \left(\frac{1}{p-1}\right)^{1/p}\|f\|_{L^p_w}$$

hold. As a consequence there exists an equivalent norm so that L^p_w is a Banach space.

PROOF. Set $A = \{|f(x)| > t\}$. Then

$$\begin{aligned} t^p|\{|f(x)|>t\}| \le \int_A t^{p-1}|f|d\mu \le & t^{p-1}\int_t^\infty \mu(\{|f|>s\})ds \\ \le & t^{p-1}\|f\|^p_{L^p_w}\int_t^\infty s^{-p}ds \\ \le & \frac{1}{p-1}\|f\|^p_{L^p_w} \end{aligned}$$

Thus, for $f > 0$

$$\begin{aligned} \|f\|^p_{L^p_w} \le & \sup t^{p-1}\mu(|f|>t)^{1/p'} \sup_A (\mu(A))^{-1/p'}\int_A f d\mu \\ \le & t^{p-1}t^{-\frac{p}{p'}}\|f\|^{\frac{p}{p'}}_{L^p_w} \sup_A(\mu(A))^{-1/p'}\int_A f d\mu \\ = & \|f\|^{p-1}_{L^p_w}\sup_A(\mu(A))^{-1/p'}\int_A f d\mu \end{aligned}$$

which yields the left inequality in the lemma after dividing by $\|f\|^{p-1}_{L^p_w}$. The inequality on the right hand side follows from the second inequality above. The quantity in the middle satisfies the triangle inequality. Completeness is easy. □

PROOF OF THEOREM 2.24. Let f be a simple function with $\|f\|_{L^p_w} = 1$. We claim that

$$\|Tf\|_{L^q_w} \le c\|f\|_{L^p_w} \tag{2.10}$$

This follows from

$$\left|\int Tf\chi_B dy\right| \le c\|f\|_{L^p_w}\nu(B)^{1/q'}$$

by Lemma 2.26

$$\|Tf\|_{L^q_w} \le c\sup_B \nu(B)^{-\frac{1}{q'}}\left|\int_B Tf\nu\right| \le c'\|f\|_{L^p_w}.$$

$$f = \sum f_j\chi_{A_j}$$

with $f_j(x) = f(x)$ if $2^j \le |f| \le 2^{j+1}$. We decompose f_j into the positive and negative part, and argue similar for both. Let $f_j \ge 0$. We can write it as

$$f_j\chi_{A_j} = \sum t_l\chi_{C_l}$$

with $t_l \ge 0$ and $C_{l+1} \subset C_l \subset A_j$ for all l. Then

$$\left|\int_B T(f_j\chi_{A_j})\nu\right| \le c\|f\|_{L^\infty}\mu(A_j)^{1/p_0}\mu(B)^{1/q_0'}.$$

since $\sum t_j \le \|f\|_{L^\infty}$. We obtain the same estimate with the index 1.

Let $\|f\|_{L^p_w} = 1$ and $s = \nu(B)$. Then

$$\begin{aligned}\left|\int_B |Tf|\nu\right| \le & c\sum_j \int_B |Tf\chi_{A_j}|\nu \\ \le & \sum_j 2^j \min\{\mu^{1/p_0}(A_j)\nu^{1/q_0'}(B), \mu^{1/p_1}(A_j)\nu^{1/q_1'}(B)\} \\ \le & c\sum \min\{2^{j-jp/p_0}s^{1/q_0'}, 2^{j-jp/p_1}s^{1/q_1'}\} \\ \le & c\left(\sum_{j\le J} 2^{j-jp/p_1}s^{1/q_1'} + \sum_{j>J} 2^{j-jp/p_0}s^{1/q_0'}\right) \\ \le & C(2^{J(1-\frac{p}{p_1})}s^{1/q_1'} + 2^{J(1-\frac{p}{p_0})}s^{1/q_0'}) \\ \le & 2Cs^{1/q'}\end{aligned}$$

if we choose J appropriately so that both summands have the same size:

$$\frac{Jp}{\ln_2 s}\left(\frac{1}{p_0} - \frac{1}{p_1}\right) = \frac{1}{q_0'} - \frac{1}{q_1'}$$

then (check if $p = p_0$ and $p = p_1$, plus linearity)

$$2^{J(1-\frac{p}{p_1}) + \frac{\ln_2 s}{q_1'}} = s^{\frac{1}{q'}}.$$

The same argument yields

$$\left|\int f\chi_A g\right| \le c\mu(A)^{1/p}\|f\|_{L^\infty}\|g\|_{L^{q'}_w}. \tag{2.11}$$

We read (2.10) and (2.11) as L^∞ resp L^1 estimates. We define the Lorentz spaces rsp. norms

$$\|f\|^q_{L^{pq}} := q\int \left(t\mu(\{x : |f(x)| > t\})^{1/p}\right)^q \frac{dt}{t}. \tag{2.12}$$

Then

$$\|f\|_{L^p_w} = \|f\|_{L^{p\infty}} \le \|f\|_{L^{pq}}$$

since

$$\begin{aligned}t\mu(\{|f(x)| > t\})^{1/p} \le & \int_0^t qs^{q-1}ds\mu(\{|f(x)| > t\})^{1/p} \\ \le & q\int_0^\infty (s\mu(\{|f| > s\})^{1/p})^q \frac{ds}{s}.\end{aligned}$$

and, if $q_1 < q_2$

$$\|f\|^{q_2}_{L^{pq_2}} \le \|f\|^{q_1}_{L^{pq_1}}\|f\|^{q_2-q_1}_{L^{p\infty}} \le \|^{q_2}_{L^{pq_1}}$$

hence

$$\|f\|_{L^{pq_1}} \le \|f\|_{L^{pq_1}}. \tag{2.13}$$

Lemma 2.27. *Let $1 < p < \infty$ and $1 \le q \le \infty$. Then the generalized Hölder inequality*

$$\left|\int fg\mu\right| \le c\|f\|_{L^{pq}}\|g\|_{L^{p'q'}} \tag{2.14}$$

holds and

$$\|f\|_{L^{pq}} \le c \sup_{\|g\|_{L^{p'q'}}\le 1} \int fg\mu. \tag{2.15}$$

Proof. To prove (2.14) for functions f and g we define monotonically decreasing functions $f^*, g^* : (0,\infty) \to [0,\infty)$ so that

$$|\{f^* > s\}| = \mu(\{|f| > s\})$$

for all $s > 0$. With this notation the Hardy-Littlewood inequality holds:

$$\int fg\mu \le \int_0^\infty f^*(t)g^*(t)dt. \tag{2.16}$$

We have

$$\begin{aligned}
\int |f(x)g(x)|\mu &= \int_X \int_0^\infty \int_0^\infty \chi_{|f(x)|\ge s}\chi_{|g(x)|\ge t} ds dt \\
&= \int_X \int_0^\infty \int_0^\infty \chi_{\{|f(x)|>s, |g(x)|>t\}} ds dt \\
&= \int_0^\infty \int_0^\infty \mu(\{|f(x)| > s, g(x) > t\}) ds\, dt \\
&\le \int_0^\infty \int_0^\infty m^1(\{f^* > s\} \cap \{|g^* > t\}) ds\, dt \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \chi_{f^*>s}(u)\chi_{g^*>t}(u) du\, dt\, ds \\
&= \int_0^\infty f^*(u)g^*(u)du.
\end{aligned}$$

Inequality (2.14) follows by Hölder's inequality from this estimate:

$$\begin{aligned}
\int fg d\mu &\le \int_0^\infty f^* g^* du \\
&\le \left(\int_0^\infty t^{q/p}(f^*(t))^q \frac{dt}{t} \right)^{1/q} \left(\int_0^\infty t^{q'/p'}(g^*(t))^{q'} \frac{dt}{t} \right)^{1/q'}
\end{aligned}$$

and, with the Riemann Stieltjes integral (or less transparently, by substitution with $s = f^*(t)$)

$$\begin{aligned}
\frac{q}{p} \int_0^\infty t^{q/p}(f^*(t))^q \frac{dt}{t} &= \int_0^\infty (f^*(t))^q dt^{q/p} \\
&= \int_0^\infty t^{q/p} d(f^*)^q \\
&= q \int_0^\infty (s\mu(\{|f| \ge s\})^{1/p})^q \frac{ds}{s}.
\end{aligned}$$

This completes the proof of (2.14).

Let $q < \infty$ and $f \in L^p(\mu)$. Then, with $g = |f|^{p/p'} \frac{f}{|f|}$

$$\int fg\mu = \|f\|_{L^p}^p, \qquad \|g\|_{L^{p'}} = \|f\|_{L^p}^{p-1}.$$

Let $f \in L^{pq}$. We write it as $f = \sum f_j \chi_{A_j}$ as above. The first Ansatz is

$$g = \sum_j \left(2^j \mu(A_j)^{1/p}\right)^{q-p} 2^{j(p/p')} \frac{f}{|f|} \chi_{A_j}.$$

Then

$$\int f g \mu \sim \|f\|_{L^{pq}(\mu)}.$$

Recall that this sum is finite, $\sum_{j=-N}^N$. For $\kappa > 0$ we set

$$g^\kappa = \sum_j \left(2^{j(1+p\kappa)} \mu(A_j)^{1/p}\right)^{q-p} 2^{j(p/p')} \frac{f}{|f|} \chi_{B_j}.$$

with $B_j \subset A_j$ and $\mu(B_j) = 2^{-\kappa(j+N)} \mu(A_j)$. A tedious calculation shows that $\|g^\kappa\|_{L^{pq}(\mu)}$ is bounded independent of the the exponents provided κ is sufficiently large.

The case $q = \infty$ follows from Lemma 2.26 and

$$\|f\|_{L^{pq}(\mu)} \sim \| \|f_j \chi_{A_j}\|_{L^p} \|_{l^q}.$$

□

We read (2.11) as

$$\|Tf\|_{L^{q1}} = \|t\nu(\{y : |Tf(y)| > t\})^{1/q}\|_{L^1} \le c\|t\mu(\{x : |f(x)| > t)^{1/p}\|_{L^1}$$

Now the real interpolation argument of for the maximal function with a small modification in the definition of f_t which yields

$$\|Tf\|_{L^{qr}} \le \|f\|_{L^{pr}}. \tag{2.17}$$

for $1 \le r \le \infty$. If $p \le q$ we set $r = p$ and conclude by

$$\|Tf\|_{L^q} \le \|Tf\|_{L^{qp}} \le c\|f\|_{L^p}.$$

The transition from simple functions to general functions is an approximation argument. It suffices to prove norm continuity on simple functions. Let $p \le q$ and $f, g \in L^p(\mu)$. The argument is similar for L^p_w. Let $h \in L^{q'}$. It suffices to show

$$|\int (Tf - Tg))h\nu| \le c\|h\|_{L^{q'}} \|f - g\|_{L^p}.$$

Let

$$\phi(f) = |\int Tfh\nu|.$$

It satisfies for $t > 0$

$$\phi(tf) = t\phi(f),$$
$$0 \le \phi(f+g) \le \phi(f) + \phi(g)$$

and

$$\phi(f) \le c\|f\|_{L^p(\mu)} \|h\|_{L^{q'}(\nu)}.$$

The continuity follows from

$$|\phi(f) - \phi(g)| \le c\|h\|_{L^{q'}(\nu)} \|f - g\|_{L^p(\nu)}.$$

Continuity on simple functions of

$$\int Tfhd\mu$$

follows from the continuity assumption of the theorem. We consider

$$\eta(t) = \int T(f + t(g - f))h d\mu$$

for $0 \le t \le 1$. If $\eta(t) = 0$ for some t then

$$\left| \int Tgh\mu \right| + \left| \int Tfh\mu \right| \le c\|h\|_{L^{q'}} \|f - g\|_{L^p}.$$

Otherwise it does not change sign, and the assertion follows again from the continuity of ϕ. □

7. Higher dimensions

The definition of the convolution and the Fourier transform carries over to several different situations.

7.1. Fourier transform on $\mathbb{T}^n$. We define the n dimensional torus by $\mathbb{R}^n \backslash \mathbb{T}^n$. Again we identify functions on $\mathbb{T}^n$ by 1 periodic functions on $\mathbb{R}^n$. We define the Fourier coefficients for $m \in \mathbb{Z}^n$ by

$$\hat{f}(m) = \int_{\mathbb{T}^n} f(x) e^{-2\pi i m \cdot x} dx.$$

Many but not all of the statements have analogues in this higher dimensional setting. In particular the functions $(e^{2\pi i m \cdot x})_m$ are an orthonormal basis.

7.2. The case of a general lattice. Let $v_i \in \mathbb{R}^n$, $1 \le i \le n$ be a basis in $\mathbb{R}^n$. Its linear combinations with integer coefficients define a lattice in $\mathbb{R}^n$. We denote it by $\mathcal{L}$. The dual lattice consists of all vectors in the dual space (which we identify with $\mathbb{R}^n$) which map the elements of the lattice $\mathcal{L}$ to $\mathbb{Z}$. We denote it by $\mathcal{L}^*$. Again we identify functions on the torus $\mathbb{R}^n/\mathcal{L}$ with $\mathcal{L}$ periodic functions. The Fourier coefficients are defined for $m \in \mathcal{L}^*$ by

$$\hat{f}(m) = \int_{\mathbb{R}^n/\mathcal{L}} e^{-2\pi i m \cdot x} f(x) dx.$$

The functions

$$|\det(v_1, \dots, v_n)|^{-1/2} e^{2\pi i m \cdot x}$$

are an orthonormal basis.

7.3. The case of $\mathbb{R}^n$. We will later look at the Fourier transform which for integrable functions in $\mathbb{R}^n$ is defined by

$$\hat{f}(\xi) = \int e^{2\pi i x \cdot \xi} f(x) dx.$$

CHAPTER 3

Harmonic functions and the Poisson kernel

1. Basic properties and the Poisson kernel

DEFINITION 3.1. *Let $U \subset \mathbb{R}^n$ be open. A two times differentiable function u is called harmonic if*

$$\Delta u = \sum_{j=0}^{n} \partial_{x_j}^2 u = 0$$

(1) The real part of holomorphic functions is harmonic.
(2) Harmonic function satisfy the mean value property. A function u is harmonic if and only if

$$u(x) = \frac{1}{n|B_1(0)|} \int_{\partial B_1(0)} u(x+ry) d\mathcal{H}^{n-1}(y)$$

whenever $\overline{B_r(x)} \subset U$. This follows by computing

$$\begin{aligned} \frac{d}{dr} \int_{\partial B_1(0)} u(x+ry) d\mathcal{H}^{n-1}(y) &= \int_{\partial B_1(0)} y \cdot \nabla_y (x+ry) d\mathcal{H}^{n-1}(y) \\ &= \int_{B_1(0)} \Delta_y (x+ry) dm^n(y) \\ &= 0 \end{aligned}$$

and a slightly refined argument gives the converse.
(3) They satisfy a maximum principle: If the maximum is assumed in the interior then a harmonic function is constant on the connected part of its domain of definition.
(4) There is the fundamental solution

$$g(x) = \begin{cases} -\frac{1}{2\pi} \ln |x| & \text{if } n = 2 \\ \frac{1}{n(n-2)|B_1(0)|} |x|^{2-n} & \text{if } n \geq 3 \end{cases}$$

and

$$-\Delta g * f = f$$

whenever $f \in C^s(\mathbb{R}^n)$ with compact support: First $\Delta g(x) = 0$ if $x \neq 0$, and then, formally

$$\partial_j g * f = -\frac{1}{n|B_1(0)|} \frac{x_j}{|x|^n} * f$$

and

$$\partial_l \int_{\mathbb{R}^n \setminus B_r(x)} f(y) \frac{x_j - y_j}{|x-y|^n} = \int_{\mathbb{R}^n \setminus B_r(x)} \frac{n(x_j - y_j)(x_l - y_l) - \delta_{jl}|x-y|^2}{|x-y|^{n+2}} f(y) dy$$
$$- r^{1-n} \int_{\partial B_r(0)} \frac{y_j y_l}{|y|^2} f(x-y) d\sigma.$$

The first term vanishes for the Laplacian Δ (if we sum over $j = l$. The second converges to $-n|B_1(0)|f(x)$ as $r \to 0$. See (Evans, PDE) for details. The Hölder regularity is needed to show existence of second derivatives.

1.1. The Greens function in the half space and the Poisson kernel. We denote x' for the first $n-1$ components of $x \in \mathbb{R}^n$ and by e_n the nth vector for the standard basis.

We define the Greens function in the half space by

$$g_H(x, y) = g(x, y) - g(x, (y', -y_n) = g_H(y, x).$$

In physical terms g is the potential of an electric field with the charge in y, with a conducting boundary at $x_n = 0$, normalized so that g vanishes in this hyperplane. This effect is obtained by putting a particle with the opposite charge at the reflected point.

If f is Hölder continuous then

$$u(x) = \int_{y_n > 0} g_H(x, y) f(y) dy$$

satisfies the inhomogeneous Dirichlet problem

$$-\Delta u = f \qquad u(x) = 0$$

Let u be a bounded continuous function on $x_m \geq 0$, harmonic in $x_n > 0$, satisfying

$$|u(x)| \leq c|x|^{-1}, |Du(x)| \leq c|x|^{-2}.$$

Then, for $\varepsilon > 0$ and $x_n > \varepsilon$, since the second part of the Poisson kernel is harmonic in H, by several applications of the divergence theorem,

(3.1)

$$
\begin{aligned}
0 =& \int_{y_n>0,|x-y|>\varepsilon} u(y)\Delta_y g_H(x,y)dy \\
=& -\int_{\mathbb{R}^{n-1}} u(y')\partial_{y_n} g_H(x,y')m^{n-1}(y') - \frac{1}{n|B_1(0)|}\int_{|y|=1} u(x+\varepsilon y)d\sigma(y) \\
& \quad - \varepsilon^{n-1}\int_{|y|=1} u(x+\varepsilon y)y\cdot\nabla g(x+\varepsilon y-\tilde{x})d\sigma(y) \\
& - \int_{y_n>0,|x-y|>\varepsilon} \nabla_y u \nabla_y g_H(x,y)dy \\
=& -\int_{\mathbb{R}^{n-1}} u(y')\partial_{y_n} g_H(x,y')m^{n-1}(y') - \frac{1}{n|B_1(0)|}\int_{|y|=1} u(x+\varepsilon y)d\sigma \\
& \quad - \varepsilon^{n-1}\int_{|y|=1} u(x+\varepsilon y)y\cdot\nabla g(x+\varepsilon y-\tilde{x})d\sigma \\
& + \frac{\varepsilon}{(n-2)n|B_1(0)|}\int_{|y|=1} y\cdot\nabla u(x+\varepsilon y)d\sigma \\
& + \varepsilon^{n-1}\int_{|y|=1} g(x+\varepsilon y-\tilde{x})y\cdot\nabla u(x+\varepsilon y)d\sigma \\
& + \int_{y_n>0,|x-y|>\varepsilon} \Delta u g_H(x,y)dy \\
\to& -u(x) + \int_{\mathbb{R}^{n-1}} u(y')P(x,y')dm^{n-1}(y')
\end{aligned}
$$

where we define the Poisson kernel by

$$P(x,y') := -\partial_{y_n} g_H(x,y') = \frac{1}{n|B_1|}\frac{2x_n}{|x-y'|^n}.$$

There is a small modification by an additional logarithmic term if $n = 2$.

We claim that

$$\frac{1}{n|B_1|}\int_{\mathbb{R}^{n-1}} \frac{2x_n}{(x_n^2+|x'-y'|^2)^{n/2}} m^{n-1}(y') = 1. \tag{3.2}$$

To see this we apply the identity above to

$$u_t(x) = t^{n-1}\partial_{x_n} g(x+te_n).$$

Then $u_t(x)$ converges to a nonzero constant as $t \to \infty$ and we obtain (3.2). Moreover

$$\left|\partial^\alpha_{x'} P(x,y')\right| \le c_\alpha x_n |x-y'|^{-n-|\alpha|}.$$

Thus $t \to \frac{2t}{(t^2+|x'|^2)^{n/2}}$ is an approximate identity (with the continuous parameter t instead of n, where $t \to 0$ corresponds to $n \to \infty$, it satisfies

1. $\sup_t \int |P(x'+te_n,0)|dx' < \infty$
2. $\int P(x'+te_n,0)dx' = 1$
3. for all $\delta > 0$

$$\int_{\mathbb{R}^{n-1}\setminus B_\delta(0)} |P(x'+te_n,0)|dx' \to 0 \qquad \text{as } t \to 0$$

and $u_t * f \to f$ almost everywhere where $u_t(x') = P(x' + te_n, 0)$ and

$$u_t * f = \int P(x' + te_n, y') f(y') dy'$$

if $f \in L^p(\mathbb{R}^{n-1})$.

LEMMA 3.2. *Suppose that $g \in C(\mathbb{R}^{n-1})$ has compact support. Let*

$$u(x) = \int_{\mathbb{R}^{n-1}} P(x, y') f(y') dy'.$$

Then u is bounded and harmonic on $x_n > 0$ and continuous up to $x_n = 0$. Moreover $u(x', 0) = g(x')$.

1.2. Green's function and Poisson kernel on the unit ball. We define the reflection at the unit sphere by

$$\mathbb{R}^n \backslash \{0\} \ni x \to \frac{x}{|x|^2} = \tilde{x}$$

and we define

$$g_B(x, y) = g(y - x) - g(|y|(x - \tilde{y}))$$

If $|x| = 1$ and $y \neq 0$

$$\begin{aligned} |y|^2 |x - \tilde{y}|^2 =& |y|^2 (|x|^2 - \frac{2y \cdot x}{|y|^2} + \frac{1}{|y|^2}) \\ =& |y|^2 + 2y \cdot x + 1 = |x - y|^2 \end{aligned}$$

and thus

$$g_B(x, y) = 0$$

if $|x| = 1$. Also

$$g_B(x, y) = g_B(y, x).$$

This has the same interpretation as for the half space. In particular, if $f \in C(\overline{B}_1(0))$ and

$$u(x) = \int_{B_1(0)} g_B(x, y) f(y) dy$$

then u it two times continuous differentiable in $B_1(0)$, $-\Delta u = f$ and $u(x) = 0$ if $|x| = 1$.

We define the Poisson kernel for $|x| < 1$ and $|y| = 1$ by

$$\begin{aligned} P(x, y) =& -\sum_i y_i \partial_i g_B(x, y) \\ =& \frac{1}{n|B_1|} \frac{1 - |x|^2}{|x - y|^n}. \end{aligned}$$

Let $g \in C(\partial B)$ and

$$u(x) = \int_{\partial B_1(0)} P(x, y) g(y) d\sigma(y)$$

is harmonic and continuous up to the boundary. Moreover

$$u(x) = g(x)$$

if $|x| = 1$. The first statement is obvious. The second statement follows as for the half space, with modifications for the approximate identity on the sphere, which we do not discuss here.

LEMMA 3.3. *Suppose that* $f \in C(\partial B_1(0))$ *and*

$$u(x) = \int_{\partial B_1(0)} P(x,y) f(y) d\sigma$$

then u *is continuous on the closure, twice continuously differentiable and it satisfies*

$$\Delta u = 0$$

and $u = f$ *on* $\partial B_1(0)$.

1.2.1. *The Poisson integral in the two dimensional case.* We define the Poisson integral in polar coordinates for $0 \le r < 1$

$$\begin{aligned} P_r(x) &= \sum_{n=-\infty}^{\infty} r^{|n|} e^{2\pi i n x} \\ &= \frac{1-r^2}{1-2r\cos 2\pi x + r^2} \\ &= \mathrm{Re}\left(\frac{1+re^{2\pi i x}}{1-r^{i2\pi i x}}\right) \end{aligned} \tag{3.3}$$

where the second identity is an evaluation of the two geometric series, and the third is an algebraic manipulation. Comparison with the kernel above shows that

$$P(re^{2\pi i x}, e^{2\pi i y}) = P_r(x-y).$$

There is an alternative derivation in this case. Let $f \in C^s(\mathbb{T})$. We expand it into a Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} \int e^{-2\pi i n y} f(y) dy \, e^{2\pi i n x}$$

If f is real then

$$\int e^{-2\pi i n y} f(y) dy = \overline{\int e^{2\pi i n y} f(y) dy}$$

and

$$\begin{aligned} u(re^{2\pi i x}) &= \sum_{n=-\infty}^{\infty} r^{|n|} \int_{\mathbb{T}} e^{2\pi i n x - y} f(y) dy \\ &= \int_{\mathbb{T}} P_r(x-y) f(y) dy \end{aligned}$$

Let

$$Q_r(x) = \mathrm{Im}\left(\frac{1+re^{2\pi i x}}{1-re^{2\pi i x}}\right) = \frac{2r\sin(2\pi x)}{1-2r\cos(2\pi x)+r^2}. \tag{3.4}$$

Then

$$P_r(x) + iQ_r(x)$$

is a holomorphic function if $z = re^{2\pi i x}$. We denote the unit disc by $\mathbb{D}$.

2. Boundary behaviour of harmonic functions

Given a real measure $\mu \in \mathcal{M}(\mathbb{T})$ we define the harmonic function

$$u(re^{2\pi ix}) = \int P_r(x-y)\mu(y).$$

We recall that

$$P_r(x) = \mathrm{Re}\left(\frac{1+re^{2\pi ix}}{1-re^{2\pi ix}}\right)$$

satisfies

1. $0 \le P_r(x)$ for $0 \le r < 1$
2. $\int P_r(x)dx = 1$ for $0 \le r < 1$
3. $\int_\delta^{1-\delta} P_r(x)dx \to 0$ as $r \to \infty$

and hence it is an approximate identity.

THEOREM 3.4. *The function u satisfies*

$$\sup_{0\le r\le 1} \int_0^1 |u(re^{2\pi ix})|dx = \lim_{r\to 1} \int_0^1 |u(re^{2\pi ix})|dx = \|\mu\|_{\mathcal{M}}.$$

Any such function determines a measure uniquely.

1. *μ is absolutely continuous with respect to the Lebesgue measure if and only if $u(re^{2\pi i \cdot})$ converges in L^1 to the density with respect to the Lebesgue measure.*
2. *Let $1 < p \le \infty$ and $\mu = fm^1$. The following assertions are equivalent:*
   - *$f \in L^p(\mathbb{T})$.*
   - $\sup_{0<r<1} \|u(re^{2\pi i\cdot})\|_{L^p(\mathbb{T})} < \infty$
   - *$u(re^{2\pi i\cdot})$ converges in L^p resp. weak $*$ in L^∞.*
3. *f is continuous if and only if u defines a continuous function on $\overline{B}_1(0)$.*

Before we turn to the proof we formulate a useful result.

LEMMA 3.5. (1) *Suppose that $F \in C(\overline{\mathbb{D}})$ is harmonic in the open disc. Then $F_r = P_r * F_1$ for $0 \le r < 1$.*
(2) *If F is harmonic in the interior then $F_{sr} = P_r * F_s$ for $0 \le r, s, < 1$.*
(3) *If F is harmonic in the interior and $1 \le p \le \infty$ then $r \to \|F_r\|_{L^p(\mathbb{T})}$ is non-decreasing.*

PROOF. Let F be as in (1). Then $P_r * F_1$ defines a continuous harmonic function (since P_r is an approximate identity as $r \to 1$). Thus $v = F(re^{2\pi ix}) - P_r * F_1(x)$ is harmonic and zero at the boundary. By the maximum principle $v = 0$. Point (2) follows by rescaling. By (2) and Young's inequality $\|F_{rs}\|_{L^p(\mathbb{T})} \le \|F_r\|_{L^p(\mathbb{T})}$. □

PROOF OF THEOREM 3.4. Let F be harmonic in $\mathbb{D}$ with

$$\sup_r \|F_r\|_{L^1(\mathbb{T})} = C < \infty. \tag{3.5}$$

The F_r defines a uniformly bounded family of real measures on $\mathbb{T}$. The space of measures $\mathcal{M}$ is dual to the separable space $C(\mathbb{T})$ of continuous functions. By the theorem of Banach Alaoglou the closed unit ball in $\mathcal{M}(\mathbb{T})$ is weak* compact, and there exists a weakly converging subsequence $F_{r_j} \to \mu$ as $r_j \to 1$. Then, for $0 < r < 1$

$$P_r * \mu = \lim_{j\to\infty} P_r * F_{r_j} = \lim_{j\to\infty} F_{r_j r} = F_r$$

and, for $f \in C(\mathbb{T})$,

$$\int f F_r dx = \int f P_r * \mu dx = \int P_r * f \mu \to \int f\mu$$

as $r \to 1$. Thus $F_r \to \mu$ weak* in $\mathcal{M}(\mathbb{T})$ and any harmonic function which satisfies (3.5) determines a measure on $\mathbb{T}$. Moreover

$$\|\mu\|_{\mathcal{M}(\mathbb{T})} \leq \liminf_{r\to 1} \|F_r\|_{L^1(\mathbb{T})}.$$

We have seen that Young's inequality implies the converse statement and hence we have equality. If $f \in L^1(\mathbb{T})$ then Proposition 2.7 implies convergence in $L^1(\mathbb{T})$. Conversely, if $T_r \to f$ in L^1 then $\mu = f dx$.

The similar statements of the next point follow by the same type of argument.

Finally, if f is continuous then $F_r = P_r * f$ defines by Proposition 2.7 a continuous function. The converse statement is obvious. □

3. Almost everywhere convergence

DEFINITION 3.6. *Let (Φ_n) by an approximate identity. We say it is radially bounded if there exists functions Ψ_n so that $|\Phi_n| \leq \Psi_n$, Ψ_n is radial and radially nonincreasing ($\psi_n(x) \geq \psi_n(y)$ if $0 \leq |x| \leq |y| \leq \frac{1}{2}$) and $\sup_n \|\Psi_n\|_{L^1} < \infty$.*

LEMMA 3.7. *Let (Φ_n) and Ψ_n be as in the definition. Then for $f \in L^1(\mathbb{T})$*

$$\sup_n |\Phi_n * f(x)| \leq \sup_n \|\psi_n\|_{L^1(\mathbb{T})} Mf(x).$$

PROOF. Let K be even and radially nonincreasing. We claim that

$$|K * f(x)| \leq \|K\|_{L^1} Mf(x).$$

Assume that this is true. Then the statement follows from

$$|\Phi_n * f(x)| \leq \psi_n * |f|(x) \leq \|\psi_n\|_{L^1} Mf(x).$$

The claim holds for even characteristic functions, and for radially nonincreasing simple functions, and by a further approximation it holds in the generality of the lemma. □

THEOREM 3.8. *Let (Φ_n) be a radially bounded approximate identity. Let $f \in L^1(\mathbb{T})$. Then $\Phi_n * f \to f$ almost everywhere as $n \to \infty$.*

PROOF. Pick $\varepsilon > 0$ and $g \in C(\mathbb{T})$ with $\|f - g\|_{L^1(\mathbb{T})} < \varepsilon/2$. Then

$$\begin{aligned} &|\{x \in \mathbb{T} : \limsup_{n\to\infty} |\Phi_n * f(x) - f(x)| > \varepsilon^{1/2}\}| \\ &\qquad \leq |\{x \in \mathbb{T} : \limsup_{n\to\infty} |\Phi_n * (f-g)(x) - (f-g)(x)| > \varepsilon^{1/2}\}| \\ &\qquad \leq 3 \sup_n \|\psi_n\|_{L^1} \varepsilon^{-1/2} \|f - g\|_{L^1} \\ &\qquad \leq 3 \sup_n \|\psi_n\|_{L^1} \varepsilon^{1/2} \end{aligned}$$

As a consequence

$$|\{x \in \mathbb{T} : \limsup_{n\to\infty} |\Phi_n * f(x) - f(x)| > 0\}| = 0.$$

□

4. Subharmonic functions

DEFINITION 3.9. *Let $U \subset \mathbb{R}^n$ be open. A function $f : U \to [-\infty, \infty$ is called subharmonic if it is continuous (with the obvious meaning for the value $-\infty$) and if for all $x \in U$ there exists $r > 0$ so that*

$$f(x) \le \frac{1}{n|B_1(0)|} \int_{\partial B_1(0)} f(x + sy) d\sigma(y).$$

for all $0 \le s < r$.

LEMMA 3.10. (1) *If f and f are subharmonic then $\max\{f, g\}$ is subharmonic.*
(2) *A function $f \in C^2(U)$ is subharmonic if and only if $\Delta f \ge 0$.*
(3) *If F is holomorphic then $\ln |F|$ and $|F|^\alpha$ for $\alpha > 0$ are subharmonic.*
(4) *If f is subharmonic and ϕ is increasing and convex then $\phi \circ f$ is subharmonic.*

PROOF. The first claim follows immediately from the definition. Subharmonicity follows from $\Delta f \ge 0$ by the argument for the mean value property:

$$\frac{d}{dr} \frac{1}{n|B_1(0)|} \int_{\partial B_1(0)} f(x - sy) d\sigma(y) = \int_{B_1(0)} \Delta f(x + y) dy.$$

It also implies the converse implication. Let x be a point where $\Delta f(x) < 0$. Then for small radii the mean value inequality is violated. We turn to the fourth point. By monotonicity and the mean value inequality resp. definition and Jensen's inequality

$$\begin{aligned} \phi(f(x)) \le& \phi \left(\frac{1}{n|B_1(0)|} \int_{\partial B_1(0)} f(x + ry) d\sigma(y) \right) \\ \le& \frac{1}{n|B_1(0)|} \int_{\partial B_1(0)} \phi \circ f(x + ry) d\sigma(y) \end{aligned}$$

If F is holomorphic then $\ln |F|$ is continuous, with values in $[-\infty, \infty)$. If $F(z_0) \neq 0$ then $\operatorname{Re} \ln F(z) = \ln |F(z)|$ is harmonic in a neighborhood and it satisfies the mean value identity near z_0. If $F(z_0) = -\infty$ there is nothing to prove. Since $|F|^\alpha = \exp(\alpha \ln |F|)$ the last statement follows from the previous points. □

LEMMA 3.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $u, v : \overline{\Omega} \to \mathbb{R}$ continuous and subharmonic resp harmonic. If $u \le v$ on $\partial\Omega$ then $u \le v$ in Ω.*

PROOF. The function $u - v$ is subharmonic, ≤ 0 at $\partial\Omega$. The definition of subharmonic function implies the maximum principle, and the maximum is assumed at the boundary. □

LEMMA 3.12. *Let $\Omega \subset \mathbb{R}^n$ be open, $f : \Omega \to [-\infty, \infty)$ be subharmonic, $\overline{B_r(x)} \subset \mathbb{R}^n$. Then*

$$f(x) \le \frac{1}{n|B_1(0)|} \int_{\partial B_1(0)} f(x+ry) d\sigma(y)$$

PROOF. Let

$$f_n(x) = \max\{f(x), -n\}.$$

It is subharmonic and it suffices to verify the assertion for f_n, or, by an abuse of notation, we assume that f is bounded. Let u be the harmonic function on $B_r(x)$ which coincides with f at the boundary. By the previous lemma $f \le u$ on $B_r(x)$, and hence

$$f(x) \le u(x) \le \frac{1}{n|B_1(0)|} \int_{\partial B_1(0)} u(x+ry) d\sigma = \frac{1}{n|B_1(0)|} \int_{\partial B_1(0)} f(x+ry) d\sigma$$

as claimed. □

The prove implies also

COROLLARY 3.13. *Suppose that f is subharmonic on the open set $U \subset \mathbb{R}^n$ and $\overline{B_r(x_0)} \subset U$. Then, if $x \in B_r(x_0)$*

$$f(x) \le \int P(\frac{x-x_0}{r}, y) f(x_0 + ry) d\sigma(y).$$

PROPOSITION 3.14. *Suppose that g is a nonnegative subharmonic function on $B_1(0) \subset \mathbb{R}^2$ which satisfies*

$$|||g|||_1 := \sup_{0<r<1} \|g(re^{2\pi i x})\|_{L^1} < \infty$$

Let

$$g^*(x) = \sup_{0 \le r < 1} |g(re^{2\pi i x})|. \tag{3.6}$$

Then

(1) $|\{x : g^*(x) > t\}| \le \frac{c}{t} |||g|||_1$
(2) *If $1 < p \le \infty$ and*

$$|||g|||_p := \sup_r \|g(re^{2\pi i x})\|_{L^p(\mathbb{T})} < \infty$$

then

$$\|g^*\|_{L^p} \le c|||g|||_p$$

PROOF. Suppose that $g_{r_j} \to \mu$ as $j \to \infty$ with $r_j \to 1$. Existence of such a sequence and μ follows from the weak* compactness argument. Then, as in the proof of Theorem 3.4 combined with Lemma 3.13

$$g(re^{2\pi i x}) \le P_r * \mu(x)$$

and

$$\lim_{r \to 1} \|g_r\|_{L^1} = \|\mu\|_{\mathcal{M}} = |||g|||_1.$$

But now

$$0 \le g(re^{2\pi i x}) \le P_r * \mu(x) \le M\mu(x)$$

and hence

$$g^*(x) \le M\mu(x)$$

This implies the weak inequality. The estimate in L^p follows in the same fashion. □

5. The theorems of Riesz

DEFINITION 3.15 (complex Hardy space). *Let* $1 \leq p \leq \infty$. *We denote the unit disc by* $\mathbb{D}$. *The space* $H^p(\mathbb{D})$ *is the space of holomorphic functions* F *on* $\mathbb{D}$ *such that*

$$\|F\|_{H^p(\mathbb{D})} = \sup_{0 \leq r < 1} \|F_r\|_{L^p(\mathbb{T})} < \infty.$$

THEOREM 3.16. *If* $F \in H^1(\mathbb{D})$ *then* $F^* \in L^1(\mathbb{T})$.

PROOF. The function $|F|^{\frac{1}{2}}$ is subharmonic and $|F_r|^{\frac{1}{2}}$ is bounded in $L^2(\mathbb{T})$, uniformly in r. By Proposition 3.14 $(|F|^{1/2})^* \in L^2$. This implies the statement of the theorem. □

By Theorem 3.4 $F_r = P_r * \mu$ for a complex measure μ on $\mathbb{T}$. By Theorem 3.16 there is the integrable majorant $|F|^*$ for the functions F_r. The functions F_r converge to the measure μ in the sense of measures. Because of the integrable majorant μ is absolutely continuous with respect the Lebesgue-measure and can be written as $\mu = fm^1$ with $f \in L^1$. Moreover $F_r = P_r * f$. But then, due to the pointwise convergence of Theorem 3.8 $F_r \to f$ as $r \to 1$ almost everywhere. We obtain the second version of the Riesz theorem.

THEOREM 3.17. *If* $F \in H^1(D)$ *and* $f = \lim_{r \to 1} F_r(x)$ *almost everywhere then* $f \in L^1(\mathbb{T})$ *and*

$$F_r(x) = P_r * f(x).$$

THEOREM 3.18. *Suppose that* $\mu \in \mathcal{M}(\mathbb{T})$ *satisfies* $\hat{\mu}(n) = 0$ *for* $n < 0$. *Then* μ *is absolutely continuous with respect to the Lebesgue measure.*

PROOF. Let F be defined by $P_r * \mu$. Then the real and imaginary part of F are harmonic. Harmonic functions are smooth and even analytic (this can be seen from the representation by the Poisson formula) and near $z = 0$ we can expand F in a Taylor expansion (by expanding the Poisson kernel) which converges in $\mathbb{D}$

$$F = \sum_{m,n \geq 0} a_{mn} z^n \bar{z}^m.$$

We apply the Laplace operator $\Delta = 4\partial_z \partial_{\bar{z}}$ with $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. Then

$$0 = \Delta F = \sum_{n,m \geq 0} a_{mn} \partial_z \partial_{\bar{z}} (z^n \bar{z}^m)$$

with $\partial_z \partial_{\bar{z}}(z^n \bar{z}^m) = nm z^{n-1} \bar{z}^{m-1}$. Thus we can write

$$F(z) = \sum_{n=0}^{\infty} a_n z^n + b_n \bar{z}^n.$$

Now we apply the assumption: For $k < 0$

$$\int e^{-2\pi i k x} \mu(x) = 0$$

and hence
$$\int e^{-2\pi ikx} P_r * \mu dx = \int P_r e^{-2\pi ikx}\mu = \int r^{|k|} e^{-2\pi ikx}\mu = 0$$
since z^{-k} is holomorphic and hence real and imaginary part are harmonic.

Thus $F \in H^1(\mathbb{D})$ and the assertion follows from Theorem 3.17. □

THEOREM 3.19. *Let $F \in H^1(\mathbb{D})$. Suppose that F is non identically* 0 *and let f be as in Theorem 3.17. Then* $\ln|f| \in L^1$. *In particular is does not vanish on a set of positive measure.*

PROOF. We first assume that $F(0) \neq 0$. Since $\ln|F|$ is subharmonic
$$\ln|F(0)| \le \int_{\mathbb{T}} \ln|F_r| dx.$$
We want to take the limit as $r \to 1$. We know that $F_r \to f$ almost everywhere and
$$\ln|F| \le |F|^*$$
is an integrable majorant for the positive part. Let $g_r = |F|^* - \ln|F_r| \ge 0$. It converges almost everywhere to $g_1 = |F|^* - \ln|f|$ and by the Lemma of fatou
$$\int g_1 dr \le \liminf_{r\to 1} \int g_1 \le \int |F|^* - \ln|F(0)|.$$
We have to remove the assumption that $F(0) \neq 0$. There exists a point $z_0 \in B_{1/2}(0)$ with $F(z_0) \neq 0$. We apply a Möbius transform to transport z_0 to 0. The Möbius transform maps circles to circles with possibly different radii and centers. For large circle the L^1 norm is bounded by $|F|^*$ before the transform.

□

This is considerable sharpening of a statement from complex analysis.

6. Conjugate functions

DEFINITION 3.20. *Let u be harmonic and real valued on $\mathbb{D}$. We define the conjugate function $\tilde u$ to be the unique harmonic function with $\tilde u(0) = 0$ such that $u + i\tilde u$ is holomorphic.*

LEMMA 3.21. *If u is constant then $\tilde u = 0$. If $f = u + iv$ is holomorphic and $v(0) = 0$ then $\tilde u = v$. If u is harmonic on $\mathbb{D}$ then*
$$\widehat{\tilde u_r}(n) = -i \operatorname{sign}(n) \widehat{u_r}(n)$$
where
$$\operatorname{sign}(n) = \begin{cases} -1 & \text{if } n < 0 \\ 0 & \text{if } n = 0 \\ 1 & \text{if } n > 0 \end{cases}$$

PROOF. The function
$$u + iv - (u + i\tilde u)$$
is purely imaginary and holomorphic, hence constant. We expand
$$u_r(x) = \sum_{k\in\mathbb{Z}} a_k(r) e^{2\pi ikx}$$

where

$$a_k = \int u(re^{2\pi ix})e^{-2\pi ikx}dx$$

and $a_k = \overline{a_{-k}}$ since u is real valued. Then

$$r^2\Delta u = \sum_{k\in\mathbb{Z}}(r^2 a_k''(r) + ra_k' - k^2 a_k(r))e^{2\pi ikx}$$

and hence $a_k(r) = a_k r^{|k|}$ and

$$u_r(x) = \sum_{k\in\mathbb{Z}} a_k r^{|k|}e^{2\pi ikx}$$

with $a_k = \overline{a_{-k}}$. Then

$$u + iv = u(0) + 2\sum_{k=1}^{\infty} a_k z^k$$

is holomorphic with real part u. Checking the Fourier coefficients gives the claimed relation of the Fourier coefficients.

□

We begin with an important weak-L^1 bound due to Besicovitch and Kolmogorov.

THEOREM 3.22. *Let u be harmonic in $\mathbb{D}$ and satisfy*

$$|||u|||_1 = \sup_{0<r<1} \|u_r\|_{L^1(\mathbb{T})} < \infty.$$

Then

$$|\{x : \tilde{u}^*(x) > t\}| \le \frac{c}{t}|||u|||_{L^1}.$$

PROOF. By Theorem 3.4 $u_t = P_t * \mu$ for a measure μ. It suffices to verify the assertion for nonnegative measures μ. Let

$$E_s = \{x : \tilde{u}^*(x) > s\}.$$

We define the holomorphic function $F = -\tilde{u} + iu$. Let

$$\omega_s(x,y) := \frac{1}{\pi}\int_{(-\infty,-s)\cup(s,\infty)} \frac{y}{(x-t)^2+y^2}dt$$

which is harmonic for $y > 0$ and nonnegative. Moreover

- $\omega_s(x,y) \ge \frac{1}{2}$ if $|x| > s$,
- $\omega_s(0,y) \le \frac{2y}{\pi s}$

since

$$\begin{aligned}\int_0^\infty \frac{y}{x^2+y^2}dx &= \int_0^\infty \frac{1}{(x/y)^2+1}\frac{dx}{y} \\ &= \int_0^\infty \frac{1}{1+z^2}dz \\ &= \arctan(\infty) - \arctan(0) \\ &= \frac{\pi}{2}\end{aligned}$$

and

$$\omega_s(0,y) = \frac{1}{\pi}\int_{(-\infty,-s)\cup(s,\infty)} \frac{y}{t^2+y^2}dt \le \frac{2}{\pi}\int_{s/y}^{\infty} \frac{dt}{1+t^2} \le \frac{2y}{\pi s}.$$

The composition $\omega \circ F$ is harmonic and, if $x \in E_s$ then $\omega(F(x)) \ge \frac{1}{2}$ for some $0 < r < 1$. By Proposition 3.14

$$|E_s| \le |\{y : (\omega_s \circ F)^* \ge \frac{1}{2}\}| \le 6|\|\omega_s F\||_1.$$

Since $\omega_s \circ F \ge 0$ the mean value property implies

$$|\|\omega_s \circ F\||_1 = \omega_s(F(0)) = \omega_s(iu(0)) \le \frac{2}{\pi}\frac{u(0)}{s} = \frac{2}{\pi}\frac{|\|u\||_1}{s}$$

which gives

$$|E_s| \le \frac{12}{\pi}\frac{|\|u\||_1}{s}.$$

□

LEMMA 3.23. *Let u be harmonic on D. Then*

$$\|\tilde{u}_r\|_{L^2}^2 + u(0)^2 = \|u_r\|_{L^2}^2$$

PROOF. This is seen by Plancherel and the relation of the Fourier coefficients. □

COROLLARY 3.24. *If $f \in L^2(\mathbb{T})$ and $u_r = P_r * f$ then the limit $g(x) = \lim_{r\to 1} \tilde{u}_r(x)$ exists almost everywhere and in L^2.*

PROOF. By the previous lemma $|\|\tilde{u}\||_2 \le |\|\tilde{u}\||_2$. By Theorem 3.4 u_r and $\tilde{u}_r$ converge in L^2 to some functions f and $\tilde{f}$ in L^2. Since $|u_r(x)| \le Mf(x)$ and $|\tilde{u}_r(x)| \le M\tilde{f}(x)$ we obtain pointwise convergence. □

7. The Hilbert transform

We formally introduce the Hilbert transform H of a measure $\mu \in \mathcal{M}(\mathbb{T})$ as

$$H\mu = \tilde{\mu} = \lim_{r\to 1} \widetilde{P_r * \mu}$$

It is not clear whether and in which sense the limit exists. It does for $\mu = fm^1$ for a square integrable f.

COROLLARY 3.25. *If u is harmonic and $|\|u\||_1 < \infty$ then*

$$g = \lim_{r\to 1} \tilde{u}_r$$

exists almost everywhere. It satisfies the weak bound

$$|\{x : |g(x)| > t\}| \le \frac{c}{t}\|f\|_{L^1}$$

PROOF. By Theorem 3.4 $u_r = P_r * \mu$. If $\mu = fm^1$ with $f \in L^2$ then the almost everywhere convergence follows from Corollary 3.24. Next we consider $\mu = fm^1$ with $f \in L^1(\mathbb{T})$. Given $\varepsilon > 0$ there exists $g \in L^2$ with $\|f-g\|_{L^2} \le \varepsilon$. Let $v_r = P_r * g$. Then, by Theorem 3.22,

$$\begin{aligned}
|E_\varepsilon| =& |\{ |\limsup_{s,r\to 1} |(u+i\tilde u)(re^{2\pi i x}) - (u+i\tilde u)(se^{2\pi i x})| > \varepsilon^{1/2}\}| \\
=& \Big| \{ |\limsup_{s,r\to 1} |(u-v+i(\tilde u - \tilde v))(re^{2\pi i x}) \\
& - (u-v+i(\tilde u-\tilde v))(se^{2\pi i x})| > \varepsilon^{1/2}\} \Big| \\
\le & \frac{c}{\varepsilon^{1/2}} \|f-g\|_{L^1} \\
\le & c\varepsilon^{1/2}.
\end{aligned}$$

Thus

$$|\{ |\limsup_{s,r\to 1} |(u+iv)(re^{2\pi i x}) - (u+iv)(se^{2\pi i x})| > 0\}| = \lim_{\varepsilon\to 0} |E_\varepsilon| = 0.$$

Let μ be a general measure on $\mathbb{T}$. We decompose into the absolutely continuous part and a singular part: There exists $f \in L^1(\mathbb{T})$ and a set B of measure 0 so that

$$\mu(A) = \int_A f dx + \mu(A\cap B).$$

The first part we have dealt with above, and we consider a Borel measure μ such that there exists B of measure 0 so that $\mu(A) = \mu(A\cap B)$. Since

$$\mu(B) = \sup_{K\subset B, compact} \mu(K)$$

for every μ measurable set, given $\varepsilon > 0$ there exists K so that $\mu(A) < \mu(K) + \varepsilon$. We define

$$\mu^K(A) = \mu(A\cap K).$$

If I is an open interval disjoint from K and if $u_t = P_t * \mu^K$ and $w = u + i\tilde u$ then u has a continous extension to the image of I on the arc of the unit circle defined by I and the real part vanishes. By the Schwarz reflection principle w has a holomorphic extension across this arc, and hence all limits exist in the complement of the compact set K of measure 0.

As in the first step we derive the existence of the limit almost everywhere.

The bound is a consequence of Theorem 3.22.

□

THEOREM 3.26. *If* $1 < p < \infty$ *then*

$$\|Hf\|_{L^p} \le \left\{ \begin{array}{ll} \tan(\frac{\pi}{2p}) & \text{if } 1 < p \le 2 \\ \cot(\frac{\pi}{2p}) & \text{if } p \ge 2 \end{array} \right\} \|f\|_{L^p}$$

PROOF. We have seen that

$$\|Hf\|_{L^2} \le \|f\|_{L^2}$$

By the interpolation theorem of Markinciewicz this implies together with the weak estimate that

$$\|Hf\|_{L^p} \le c_p \|f\|_{L^p}$$

for $1 < p \le 2$. Since

$$\int Hfg dx = -\int fHg dx$$

the estimate (up to the sharp constant) for $p \geq 2$ follows by duality.

We provide a second proof following Grafakos' variant of Pichorides' proof, which yields a sharp constant.

LEMMA 3.27. *For a, b real and $1 < p \leq 2$*

$$|b|^p \leq \tan(\frac{\pi}{2p})|a|^p - B_b \operatorname{Re}[(|a| + ib)^p]$$

with

$$B_b = \frac{\sin^{p-1}(\frac{\pi}{2p})}{\sin(\frac{(p-1)\pi}{2p})}$$

We set $a = f(x)$ and $b = Hf(x)$ and integrate. Then

$$\int_0^1 |Hf(x)|^p dx \leq \tan^p(\frac{\pi}{2p}) \int_0^1 |f(x)|^p dx - B_p \int_0^1 \operatorname{Re}[(|f(x)| + iHf(x))^p] dx$$

and the theorem follows once we show that

$$\int \operatorname{Re}[(|f(x)| + iHf(x))^p] dx \geq 0.$$

Let $u + iv$ be the harmonic (holomorphic) extensions of $f + iHf$.

LEMMA 3.28. *The function $g(x, y) = \operatorname{Re}((|x| + iy))^p)$ is subharmonic and $g(u, v)$ is also subharmonic.*

Then

$$0 \leq g(u(0), v(0)) \leq \int_0^1 \operatorname{Re}[(|f(x)| + iHf(x))^p] dx$$

which concludes the proof up to a duality argument.

□

Sharpness follows by taking

$$f = \operatorname{Re}\left(\frac{1+z}{1-z}\right)^s$$

for suitable s.

The Hilbert transform is not bounded in L^∞: The real part of $i \ln_C(1 - e^{2\pi i x})$ is $\pi(x - \frac{1}{2})$ for $0 < x < 2\pi$ and the imaginary part is $\ln|1 - e^{2\pi i x}|$, which is not bounded.

We recall that

$$\operatorname{Im} \frac{1 + e^{2\pi i x}}{1 - e^{2\pi i x}} = \frac{\cos \pi x}{\sin \pi x} = \cot(\pi x).$$

PROPOSITION 3.29. *If $f \in C^1(\mathbb{T})$ then*

$$f(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \cot(\pi(x - y)) f(y) dy$$

PROOF. By symmetry

$$\int_{|x-y|>\varepsilon} \cot(\pi(x - y) f(y) dy = \frac{1}{2} \int_{|x-y|>\varepsilon} \cot(\pi(y)(f(x - y) - f(x + y)) dy$$

and the second integrand is bounded. □

PROPOSITION 3.30. *Suppose that* $\|f\|_{L^\infty} \leq 1$. *Then*

$$\int_0^1 e^{\alpha|Hf(x)|}dx \leq \frac{2}{\cos\alpha}$$

for $0 \leq \alpha < \frac{\pi}{2}$.

PROOF. Let f be as in the proposition. Let u be the harmonic extension of f and $F = \tilde{u} - iu$. Then by the maximum principle $|u| \leq 1$ and $\cos(\alpha u) \geq \cos\alpha$. Hence

$$\operatorname{Re} e^{\alpha f} = \operatorname{Re} e^{\alpha\tilde{u}} e^{-i\alpha u} = \cos(\alpha u) e^{\alpha\tilde{u}} \geq \cos\alpha e^{\alpha\tilde{u}}$$

and with the mean value property

$$\int_{\mathbb{T}} \operatorname{Re} e^{\alpha F_r(x)} dx = \operatorname{Re} e^{-i\alpha(u(0))} = \cos\alpha u(0) \leq 1$$

and therefore

$$\int_{\mathbb{T}} e^{\alpha\tilde{u}_r(x)} dx \leq \frac{1}{\cos\alpha}.$$

The estimate follows now by the Lemma of Fatou, and the same argument for $-f$. □

THEOREM 3.31. *Let* S_N *be the partial sum of Fourier series and* $1 < p < \infty$. *Then*

$$\sup_N \sup_{\|f\|_{L^p(T)}} \|S_N f\|_{L^p(\mathbb{T})} < \infty$$

and for all $f \in L^p(\mathbb{T})$

$$S_N f \to f \qquad \text{in } L^p \qquad \text{as } N \to \infty.$$

PROOF. We write (which we check by checking the effect on the Fourier coefficients)

$$\begin{aligned} S_N f =& e^{-2\pi i N x} \frac{1-iH}{2} \left[e^{2\pi i(2N)x} \frac{1+iH}{2} \left(e^{-2\pi i N x} f \right) \right] \\ &+ \frac{1}{2}(a_N e^{2\pi i N x} + a_{-N} e^{2\pi i N x}) \end{aligned}$$

and obtain by the boundedness of the Hilbert transform

$$\sup_N \|S_N f\|_{L^p(\mathbb{T})} \leq c_p \|f\|_{L^p(\mathbb{T})}.$$

This implies the assertion by Proposition 2.9. □

Pointwise convergence is true but it is considerably harder, and much more recent.

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CHAPTER 4

The Fourier transform on $\mathbb{R}^n$

1. Definition and first properties

DEFINITION 4.1. *The Fourier transform of a complex valued function* $f \in L^1(\mathbb{R}^n)$ *is*

$$\hat{f}(\xi) = \int e^{-2\pi i \xi \cdot x} f(x) dx.$$

Properties:

(1) $\hat{f}$ is a bounded continuous function which converges to 0 as $|\xi| \to \infty$. Moreover
$$|\hat{f}(\xi)| \le \|f\|_{L^1(\mathbb{R}^n)}.$$

(2) If $f \in L^1(\mathbb{R}^n)$, $h, \eta \in \mathbb{R}^n$ and $f_h(x) = f(x-h)$ then
$$\hat{f}_h(\xi) = e^{-2\pi i h \cdot y} \hat{f}(\xi)$$
and
$$\widehat{e^{-2\pi i \eta x} f} = \hat{f}(\xi + \eta)$$

(3) If $f, g \in L^1(\mathbb{R}^n)$ then $\widehat{f * g} = \hat{f}(\xi)\hat{g}(\xi)$ and by Fubini
$$\int f \hat{g}(\xi) d\xi = \int \hat{f}(x) g(x) dx.$$

(4) If $A : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map then
$$\widehat{f \circ A}(\xi) = |\det A|^{-1} \hat{f}(A^{-t}\xi).$$

LEMMA 4.2.
$$\exp\widehat{(-\pi|x|^2)}(\xi) = \exp(-\pi|\xi|^2)$$

PROOF. We calulate more than we need.

$$\begin{aligned} I_n = \int e^{-\pi|x|^2} dm^n = \int_0^1 |\{x : |x| \le \left(\frac{-\ln(t)}{\pi}\right)^{1/2}\}| dt \\ =& |B_1(0)|\pi^{-n/2} \int_0^1 |\ln(t)|^{n/2} dt \\ =& |B_1(0)|\pi^{-n/2} \int_0^\infty s^{\frac{n}{2}} e^{-s} ds \\ =& |B_1(0)|\pi^{-n/2} \Gamma(\frac{n+2}{2}). \end{aligned}$$

In particular

$$I_2 = \Gamma(2) = 1$$

Alternatively, by Fubini

$$I_{n+m} = I_n I_m$$

and hence

$$I_n = 1.$$

As a consequence

$$m^n(B_1(0)) = \frac{\pi^{n/2}}{\Gamma(\frac{n+2}{2})} \tag{4.1}$$

and

$$\Gamma(\frac{1}{2}) = \frac{1}{2}\Gamma(3/2) = \pi^{1/2}.$$

Let $n = 1$, $f(x) = e^{-\pi|x|^2}$ and

$$\int e^{-\pi x^2 - 2\pi i \xi x} dx = e^{-\pi|\xi|^2} \int e^{-\pi(x-i\xi)^2} dx.$$

We denote

$$J(t) = \int e^{-\pi(x-it)^2} dx.$$

Then

$$\begin{aligned} \frac{d}{dt} J(t) = & \int e^{-\pi(x-it)^2} (2\pi i)(x - it) dx \\ = & - i \int \frac{d}{dx} e^{-\pi(x-it)^2} dx \\ = & 0 \end{aligned}$$

This implies the formula in the one dimensional case. The higher dimensional case follows by Fubini. □

An example:

$$\int_{\mathbb{R}} e^{2\pi i x \xi} \frac{1}{1+x^2} dx = \pi e^{-2\pi|\xi|} \tag{4.2}$$

by the residue theorem since

$$\frac{1}{x^2+1} = \frac{1}{2i} \left(\frac{1}{x+i} - \frac{1}{x-i} \right).$$

2. The Fourier transform of Schwartz functions

DEFINITION 4.3. *The space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ on $\mathbb{R}^n$ is the space of all smooth functions for which the seminorms*

$$\sup |x^\alpha \partial^\beta f(x)|$$

are bounded for all multiindices.

Properties :

1. If $f \in \mathcal{S}$ and $g \in C^\infty$ with derivatives bounded by polynomials then $fg \in \mathcal{S}$.
2. If $f \in \mathcal{S}$ and $h, \eta \in \mathbb{R}^n$ then $f_h(x) = f(x-h) \in \mathcal{S}$ and $e^{2\pi i \eta \cdot x} f \in \mathcal{S}$.
3. $\mathcal{S} \subset L^1(\mathbb{R}^n)$ and the Fourier transform is defined for functions in $\mathcal{S}$.
4. $\widehat{(-2\pi i) x_j f} = \partial_{\xi_j} \hat{f}$
5. $\widehat{\partial_{x_j} f} = 2\pi i \xi_j \hat{f}$

(6) It follows that, if $f \in \mathcal{S}$,

$$\xi^\alpha \partial_\xi^\beta \hat{f} = (2\pi)^{|\beta|-|\alpha|}(-i)^{|\alpha|+|\beta|}\widehat{\partial_x^\alpha x^\beta f}$$

is a bounded continuous function since it is the Fourier transform of a Schwartz function.

The inversion theorem is a consequence:

THEOREM 4.4. *If $f, g \in \mathcal{S}(\mathbb{R}^n)$ then*

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\hat{f}(\xi)d\xi \tag{4.3}$$

$$\int_{\mathbb{R}^n} f\overline{g}dx = \int_{\mathbb{R}^n} \hat{f}\overline{\hat{g}}d\xi. \tag{4.4}$$

In particular the Fourier transform defines a unitary map on $L^2(\mathbb{R})$.

We denote the inverse transform by $\check{f}$.

PROOF. The family of functions

$$\phi_t = t^n e^{-\pi|tx|^2}$$

is an approximate identity as $t \to \infty$. Its Fourier transform is

$$\hat{\phi}_t = e^{-\pi\frac{|\xi|^2}{t^2}}$$

Thus, by the dominated convergence theorem and Fubini

$$\begin{aligned}
\int e^{2\pi i x\cdot\xi}\hat{f}(\xi)d\xi &= \lim_{t\to\infty}\int e^{2\pi i x\xi}e^{-\pi\frac{|\xi|^2}{t^2}}\hat{f}(\xi)d\xi \\
&= \lim_{t\to\infty}\int\int e^{2\pi i(x-y)\xi}e^{-\pi\frac{|\xi|^2}{t^2}}d\xi f(y)dy \\
&= \lim_{t\to\infty}\int t^n e^{-\pi t^2|x-y|^2}f(y)dy \\
&= f(x).
\end{aligned}$$

Now, since $\int f\hat{g} = \int \hat{f}g$ and $\overline{\hat{g}} = \check{\overline{g}}$,

$$\begin{aligned}
\int f\overline{g}d\xi &= \int f\overline{\check{\hat{g}}}d\xi \\
&= \int f\hat{\overline{\hat{g}}}dx \\
&= \int \hat{f}\overline{\hat{g}}d\xi
\end{aligned}$$

□

We obtain

PROPOSITION 4.5. *The Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ to itself. It is invertible and it satisfies*

$$\widehat{f * g} = \hat{f}\hat{g} \tag{4.5}$$

$$\widehat{fg} = \hat{f} * \hat{g} \tag{4.6}$$

Proof. Only the last statement has to be shown. We apply the inverse Fourier transform to the right hand side of the second equality. The statement then follows from the first equality. □

2.1. The Fourier transform of complex Gaussians. Let $A = A_r + iA_i$ be a symmetric invertible complex matrix with A_r positive definite. The function

$$f(x) = \exp(-\pi x^t Ax)$$

is a Schwartz function. We want to compute its Fourier transform. We write $A_r = B^t B$ and $y = Bx$. Then, with $g(y) = \exp(-\pi|y|^2 - i\pi y^t Cy)$ with $C = B^{-t} A_i B$ we have

$$\hat{f}(\xi) = |\det B|^{-1} \hat{g}(B^{-t}\xi).$$

If $A_i = 0$ we know the Fourier transform of g and obtain

$$\widehat{\exp(-\pi x^t A_r x)} = (\det A_r)^{-1/2} \exp(-\pi \xi^t A_r^{-1} \xi). \tag{4.7}$$

Lemma 4.6. *The Fourier transform of*

$$\exp(-\pi x^t Ax)$$

is, by an abuse of notation

$$(\det A)^{-1/2} \exp(-\pi \xi^t A^{-1} \xi). \tag{4.8}$$

where $(\det A)^{-1/2}$ *denotes the product of the square roots of the eigenvalues of* A.

The eigenvalues have positive real part, and the square root is uniquely defined. This is not necessarely true for the determinant. It is both remarkable and important that both f and the formula of its Fourier transform are well defined for invertible complex matrices with nonnegative real part.

Proof. By the considerations above it suffices to prove the statement for $A = 1 + iC$. By the Schur decomposition we can also diagonalise A and it suffices to verify the formula in one dimension. Let

$$h(\tau, \xi) = (1 + i\tau)^{1/2} e^{\pi(1+i\tau)^{-1}\xi^2} \int e^{-2\pi i\xi x - \pi(1+i\tau)x^2} dx$$

We differentiate with respect to ξ:

$$\begin{aligned}
\frac{dh}{d\xi} =&(1 + i\tau)^{1/2} e^{\pi(1+i\tau)^{-1}\xi^2} \int [-\frac{2\pi i\xi}{1 + i\tau} - 2\pi i x] e^{-2\pi i\xi x - \pi(1+i\tau)x^2} dx \\
=& -(1 + i\tau)^{-1/2} e^{\pi(1+i\tau)^{-1}\xi^2} \int \frac{d}{dx} e^{-2\pi i\xi x - \pi(1+i\tau)x^2} dx \\
=&0
\end{aligned}$$

hence $h(\tau, \xi) = h(\tau, 0)$ and

$$\begin{aligned}
\frac{dh(\tau, 0)}{d\tau} =&i(1 + i\tau)^{\frac{1}{2}} \int [\frac{1}{2(1 + i\tau)} - \pi x^2] e^{-\pi(1+i\tau)x^2} dx \\
=&\frac{i}{4\pi}(1 + i\tau)^{-\frac{3}{2}} \int \frac{d}{dx^2} e^{-\pi(1+i\tau)x^2} dx \\
=&0
\end{aligned}$$

This completes the proof since $h(0, 0) = 1$. □

3. The Fourier transform on tempered distributions

The space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ is the topological dual of $\mathcal{S}$, or, in different words, the set of all continuous linear maps from $\mathcal{S}$ to $\mathbb{C}$.

DEFINITION 4.7. *$T : \mathcal{S} \to \mathbb{C}$ is a tempered distribution if*

(1) *T is linear*
(2) *T is continuous, i.e. there exist N and C so that*

$$|T(f)| \le C \sup_{|\alpha|\le N, |\beta|\le N, x} |x^\alpha \partial^\beta f|$$

Let μ be a measure. It defines a distribution by

$$T_\mu(f) = \int f d\mu.$$

Let $1 \le p \le \infty$. Then $g \in L^p$ defines a tempered distribution by

$$T_g(f) = \int f g dx$$

We often identity f and T_f. It is important that the formulas do not lead to ambiguities when several interpretations are possible.

Let $\phi \in C^\infty$ with (polynomially) bounded derivatives. We define

$$(\phi T)(f) = T(\phi f)$$
$$(\partial_{x_j} T)(f) = -T(\partial_{x_j} f)$$

and

$$(\phi * T)(f) = T(\tilde{\phi} * f)$$

where $\tilde{\phi}(x) = \phi(-x)$. Then $\phi * T$ is the smooth (C^∞) function

$$\phi * T(x) = T(\phi(x - .))$$

since this expression is continuously differentiable, and

$$\phi * T(f) = T(\tilde{\phi} * f) = T(\int \phi(x-y) f(x) dx) = \int T(\phi(x - .)) f(x) dx.$$

The last inequality would be obvious for sums, and it is verified by writing the integral as a limit of sums.

Every Schwartz function defines a tempered distribution. Test functions are dense in the space of tempered distribution in the sense that, given a tempered distribution T there exists a sequence of functions $\phi_j \in C_0^\infty$ such that

$$\int \phi_j f dx \to T(f)$$

for all $\phi \in \mathcal{S}$: Let $\eta \in C_0^\infty$ have integral 1. Then

$$T(t^{-n}\phi(t.) * \phi(x/t) f) = \int T(t^{-n}\phi(t(x - .)))\phi(x/t) f(x) dx \to T(f).$$

DEFINITION 4.8 (Support of a distribution). *The support of the distribution T is defined as complement of the union of all open sets U with the property that $T(\phi) = 0$ if ϕ is supported in U.*

LEMMA 4.9. *Let T be a distribution with compact support. The convolution with T defines a continuous linear map on $\mathcal{S}$.*

We will not go much into the notion of continuity here. One way of formulating it: For all $N > 0$ there exists M and C such that

$$\sup_{|\alpha|+|\beta|\le N} |x^\alpha \partial_x^\beta T * \phi| \le C \sup_{|\alpha|+|\beta|\le M} |x^\alpha \partial^\beta \phi|.$$

PROOF. We assume that the support of T is contained in the ball of radius R. Then there exists N so that

$$|T(\phi)| \le C_N \sup_{|\alpha|\le N, |x|\le R} |\partial^\alpha \phi|$$

and

$$T * \phi(x) = T(\phi(x - .))$$

satisfies

$$|T * \phi(x)| \le C_N \sup_{|\alpha|\le N, |x-y|\le R} |\partial^\alpha \phi(y)| \le C(1 + (|x| - R)_+)^{-M}$$

for all M. Moreover $\partial_{x_i}(T * \phi) = T * (\partial_{x_i}\phi)$. □

Let T be a tempered distribution and S a compactly supported distribution. We define $\tilde{S}(\phi) = S(\tilde{\phi})$ and

$$S * T(\phi) = T * S(\phi) = T(\tilde{S} * \phi) = T(S(\phi(.-))).$$

DEFINITION 4.10. *The Fourier transform of the tempered distribution T is defined by*

$$\hat{T}(\phi) = T(\hat{\phi})$$

Properties:

- Since we identify functions with tempered distributions it is important that this does not introduce ambiguities:

$$\hat{T}_f(\phi) = T_f(\hat{\phi}) = \int f\hat{\phi}dx = \int \hat{f}\phi dx \tag{4.9}$$

- $\widehat{S * T} = \hat{S}\hat{T}$ whenever both sides satisfy the conditions for convolutions resp. products.
- Similarly $\widehat{ST} = \widehat{S} * \widehat{T}$.

DEFINITION 4.11. *A distribution is called homogeneous of degree $m \in \mathbb{C}$ if*

$$T(\phi) = \lambda^{m+n} T(\phi(\lambda x))$$

for $\lambda > 0$.

An homogeneous distribution is tempered (exercise)

LEMMA 4.12. *The Fourier transform of a homogeneous distribution of degree $-m$ is a homogeneous distribution of degree $-m-n$.*

PROOF.

$$\hat{T}(\phi) = T(\hat{\phi}) = T(\lambda^{-m-n}\hat{\phi}(./\lambda)) = \hat{T}(\lambda^{-m}\phi(\lambda(x)).$$

□

Examples:

$$\hat{1}(\phi) = \int \hat{\phi}(\xi)d\xi = \phi(0) = \delta_0(\phi) \tag{4.10}$$

$$\hat{\delta}_0(\phi) = \delta_0(\hat{\phi}) = \hat{\phi}(0) = \int \phi dx = \int \delta_0(e^{-2\pi i x\xi})\phi(\xi)d\xi \tag{4.11}$$

The same proof as for Fourier series shows

THEOREM 4.13 (Hausdorff-Young inequality). *Let* $1 \leq p \leq 2$ *and* $1/p + 1/p' = 1$. *Then*

$$\|\hat{f}\|_{L^{p'}} \leq \|f\|_{L^p}$$

and

LEMMA 4.14 (Riemann-Lebesgue). *Let* $f \in L^1$. *Then* $\hat{f}$ *and* $\check{f}$ *are continuous and*

$$\lim_{\xi\to\infty} \hat{f}(\xi) = 0$$

PROOF. Let $f_j \in C_0^\infty(\mathbb{R}^n)$ with $\|f - f_j\|_{L^1} < 1/j$.Then

$$\|\hat{f} - \hat{f}_j\|_{sup} \leq 1/j$$

and $f_j \to f$ uniformly. The assertion holds for f_j since $f_j \in \mathcal{S}$, and hence for f by uniform convergence. □

3.1. The Fourier transform of $1/x$**.** Let $f(x) = 1$ if $x > 0$ and 0 otherwise. Then formally

$$\int_0^\infty e^{-2\pi i x\xi}dx = \frac{1}{2\pi i\xi}.$$

How do we define the right hand side? There are at least four possibilities

$$\frac{1}{2\pi i x}(\phi) = \frac{1}{2}\int \frac{1}{2\pi i x}(\phi(x) - \phi(-x))dx \tag{4.12}$$

$$\frac{1}{2\pi i x + i0} = \lim_{t\to 0, t>0} \frac{1}{2\pi i x + t} \tag{4.13}$$

$$\frac{1}{2\pi i\xi - i0} = \lim_{t\to 0, t<0} \frac{1}{2\pi i\xi - t} \tag{4.14}$$

and as principle value

$$\frac{1}{2\pi i\xi}(\phi) = \lim_{\varepsilon\to 0}\int_{|\xi|>\varepsilon} \frac{1}{2\pi i\xi}(\phi(\xi) - \phi(-\xi))d\xi \tag{4.15}$$

It is an exercise to compare the different definitions and to choose the correct one.

3.2. Gaussians, heat and Schrödinger equation. We consider the heat equation

$$u_t - \Delta u = 0$$

We make the Ansatz that u and u_t are tempered distributions in x uniformly in time. A Fourier transform in x leads to

$$\hat{u}_t + (2\pi)^2|\xi|^2\hat{u} = 0$$

and

$$\hat{u}(t,\xi) = e^{-4\pi^2 t|\xi|^2}\hat{u}(0,\xi)$$

for $t > 0$. The inverse Fourier transform of $e^{-4\pi^2 t|\xi|^2}$ is

$$g_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

and

$$u(t,x) = g_t * u(0,.)(x)$$

for $t > 0$.

Similarly we deal with the Schrödinger equation

$$i\partial_t u + \Delta u = 0$$

and obtain with

$$g_t(x) = \left((4\pi i t)^{1/2}\right)^{-n} e^{\frac{|x|^2}{4it}}$$

formally

$$u(t) = g_t * u_0.$$

We denote the map $u_0 \to u(t)$ by $S(t)u_0$. It is a unitary group:

(1) $S(t+s) = S(t)S(s)$ for $s,t \in \mathbb{R}$, $S(0) = 1$.
(2) For all $u_0 \in L^2$ the map $t \to S(t)u_0 \in L^2$ is continuous.

LEMMA 4.15. *Let $1 \le p \le 2$. Then*

$$\|S(t)u_0\|_{L^{p'}} \le (4\pi|t|)^{-n(\frac{1}{p}-\frac{1}{2})}\|u_0\|_{L^p} \tag{4.16}$$

for all $u_0 \in L^p$. Suppose that $u_0 \in L^2$ and $u(t,x) = S(t)u_0(x)$. Then the support of the space time Fourier transform of u is the 'characteristic set'

$$\{(\tau,\xi) : \tau = -2\pi|\xi|^2\}$$

and

$$\hat{u}(\psi) = \int_{\mathbb{R}^n} \hat{\psi}(-2\pi|\xi|^2,\xi)d\xi$$

The estimate follows as for Hausdorff Young. The remaining part is an exercise.

LEMMA 4.16 (Strichartz estimates for the Schrödinger equation). *Let $p' = \frac{2(n+2)}{n}$. If $f \in L^p(\mathbb{R}\times\mathbb{R}^n)$ and $u_0 \in L^2$ then the solution to*

$$i\partial_t u + \Delta u = f, \qquad u(0,x) = u_0(x)$$

given by Duhamel's formula satisfies

$$\sup_t \|u(t)\|_{L^2(\mathbb{R}^n)} + \|u\|_{L^{p'}(\mathbb{R}\times\mathbb{R}^n)} \le c\left(\|u_0\|_{L^2} + \|f\|_{L^p(\mathbb{R}\times\mathbb{R}^n)}\right).$$

PROOF. Formally

$$\| \int_{s<t} S(t-s)f(s)ds\|_{L^{p'}(\mathbb{R}^n)} \le (4\pi)^{-n/2} \int_{-\infty}^t |t-s|^{-\frac{n}{n+2}} \|f(s)\|_{L^p(\mathbb{R}^n)}$$

and by the weak Young inequality

$$\|S(t-s)f(s)\|_{L^{p'}(\mathbb{R}\times\mathbb{R}^n)} \le c\|f\|_{L^p(\mathbb{R}\times\mathbb{R}^n)}.$$

Let T^* be the map from $L^p((-\infty,0)\times\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ which maps

$$f \to \int_{-\infty}^0 S(-s)f(s)ds.$$

On the Fourier side we see that

$$TT^*f(t) = i\int_{-\infty}^0 S(t-s)f(-s)ds$$

and

$$\begin{aligned}\|T\|^2_{L^2\to L^{p'}((0,\infty)\times\mathbb{R}^n)} =&\|T^*\|^2_{L^p((0,\infty)\times\mathbb{R}^n)\to L^2}\\ =&\|TT^*\|_{p((0,\infty)\times\mathbb{R}^n)\to L^{p'}((0,\infty)\times\mathbb{R}^n)}\end{aligned}$$

which is bounded by the first step. □

Remark: Restriction theorem.

3.3. The Poisson summation formula. We denote the Dirac measure at the point x by δ_x. Let

$$T = \sum_{k\in\mathbb{Z}^n} \delta_k$$

resp

$$T(f) = \sum_{k\in\mathbb{Z}^n} f(k).$$

THEOREM 4.17. *The Fourier transform of T is T.*

This statement is by the definition of the Fourier transform of distributions equivalent to the Poisson summation formula

$$\sum_{k\in\mathbb{Z}^n} f(k) = \sum_{k\in\mathbb{Z}^n} \hat f(k)$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

PROOF. If $f \in \mathcal{S}$ then

$$F(x) = \sum_{k\in\mathbb{Z}^n} f(x-k)$$

is a smooth periodic function, hence, if $m \in \mathbb{Z}^n$,

$$\begin{aligned}\hat F(m) =& \sum_{k\in\mathbb{Z}^n} \int_{[0,1)^n} f(x-k)e^{-2\pi imx}dx\\ =&\int f(x)e^{-2\pi imx}dx\\ =&\hat f(m)\end{aligned}$$

and

$$F(x) = \sum_{m\in\mathbb{Z}^n} \hat{f}(m)e^{2\pi imx}$$

which is the Poisson summation formula. □

3.4. Homogeneous distributions and the Laplace operator. We call T radial if for every $f \in \mathcal{S}$ and any orthogonal matrix O the identity $T(f) = T(f \circ O)$ holds. The Fourier transform of a radial distribution is radial.

LEMMA 4.18. *let $0 < \operatorname{Re} m < n$. Then the Fourier transform of*

$$\frac{\pi^{m/2}}{\Gamma(m/2)}|x|^{m-n} \qquad \textit{is} \qquad \frac{\pi^{(n-m)/2}}{\Gamma((n-m)/2)}|\xi|^{-m}$$

PROOF. We evaluate using polar coordinates and that the $n-1$ dimensional Hausdorff measure of the unit sphere is $n|B_1| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ for $m > -n$

$$\begin{aligned}\int |x|^m e^{-\pi|x|^2}dx =&\frac{\pi^{\frac{n}{2}-1}}{\Gamma(n/2)}\int_0^\infty r^{m+n-2}e^{-\pi r^2}2\pi r dr\\ =&\frac{1}{\pi^{m/2}\Gamma(n/2)}\int_0^\infty s^{\frac{m+n}{2}-1}e^{-s}ds\\ =&\frac{\Gamma((n+m)/2)}{\pi^{m/2}\Gamma(n/2)}\end{aligned}$$

Let T be homogeneous radial distribution of homogeneity $m \in \mathbb{C}$. We claim that there exists a radial function t of homogeneity m so that

$$T(\phi) = \int t\phi dx$$

whenever $\phi \in C_0^\infty(\mathbb{R}\backslash\{0\})$. An incorrect proof is given by choosen a smooth cutoff function η supported in the interval $B_{1/2}(0)$ with integral 1 and calculate for the n-th unit vector

$$\begin{aligned}t(e_n) =&T(\delta_{e_n})\\ =&\int_{B_{1/2}(0)} |e_n - y|^{m+n}\phi(y)T(\delta_{e_n-y})dy\\ =&T(\int_{B_{1/2}(0)} |e_n - y|^{m+n}\phi(y)\delta_{e_n-y}dy)\\ =&T(\phi(e_n + .)|.|^{m+n}).\end{aligned}$$

To do it correctly one has to apply this argument to an approximate identity. It is clear that t is radial and of homogeneity m.

If $\operatorname{Re} m > -n$ then this function is locally integrable and the identity holds for all functions in $\mathcal{S}$, which is seen by a smooth truncation argument and rescaling.

Now $|x|^{-m}$ for $m < n$ defines a homogeneous radial distribution of degree $-m$. Its Fourier transform is a radial distribution of degree $m-n$. By the considerations above it is given by a radial function of degree $m-n$, hence $c_m|x|^{m-n}$. To determine the constant we use the first step:

$$\begin{aligned}\Gamma((n-m)/2)\pi^{m/2} =& \Gamma(n/2)\int |x|^{-m}e^{-\pi|x|^2}dx \\ =& \Gamma(n/2)\int c_m|\xi|^{m-n}e^{-\pi|\xi|^2} \\ =& c_m\Gamma(m/2)\pi^{(n-m)/2}\end{aligned}$$

□

We apply these considerations to the Laplace operator for $n \geq 3$. Then, if u and f are tempered distributions

$$-\Delta u = f$$

implies

$$4\pi^2|\xi|^2\hat{u} = \hat{f}$$

and formally

$$\hat{u} = (4\pi^2)^{-1}\frac{\hat{f}}{|\xi|^2}$$

The function

$$\frac{1}{|\xi|^2}$$

is locally integrable and defines a homogeneous distribution of degree -2. Its inverse Fourier transform is

$$\frac{\pi^{2-n/2}}{\Gamma((n-2)/2)}|x|^{2-n}$$

and hence

$$u = g * f$$

with

$$g(x) = \frac{\Gamma((n+2)/2)}{n(n-2)\pi^{n/2}}|x|^{2-n}.$$

Moreover the inverse Fourier transform of

$$\frac{i\xi_j}{4\pi^2|\xi|^2}$$

is

$$-\frac{\pi^{n/2-1}}{2n\Gamma((n+2)/2)}\frac{x_j}{|x|^n}$$

which remains true for $n = 2$.

4. Oscillatory integrals and stationary phase

We begin with a study of complex Gaussian integrals. The first lemma contains the algebraic part.

PROPOSITION 4.19. *Let $p(x)$ be a polynomial in $\mathbb{R}^n$. and let $A = A_r + iA_i$ be symmetric with A invertible and inverse (a_{ij}) and A_r positive semi definite. Then*

$$\int p(x)e^{-\pi x^t Ax}dx = (\det A)^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{1}{(4\pi)^k k!} \left(\sum_{i,j=1}^{n} a_{ij}\partial^2_{ij} \right)^k p(x)\Bigg|_{x=0} \tag{4.17}$$

If A_r is not positive definite we understand both sides as the limit as $\varepsilon \to 0$ for $A + \varepsilon 1$.

PROOF. It suffices to consider A_r positive definite since both sides are continuous in A as long as A is invertible. By a change of coordinates we reduce the assertion to diagonal matrices A. It suffices to prove it for monomials, since polynomials are sums of monomials. An application of the theorem of Fubini reduces it to the one dimensional case. (Check carefully. There are binomial coefficients on the right hand side). So we want to evaluate for $\operatorname{Re}\mu > 0$

$$\int_{-\infty}^{\infty} x^m e^{-\pi\mu|x|^2}dx$$

Both sides vanish if m is odd, and it remains to prove

$$\int_{-\infty}^{\infty} x^{2k} e^{-\pi\mu|x|^2}dx = \mu^{-1/2-k}\frac{1}{(2\pi)^k k!}(2k)!$$

which in turn follows by induction on k

$$\begin{aligned}
\int_{-\infty}^{\infty} x^{2(k+1)} e^{-\pi\mu|x|^2}dx &= -\frac{1}{2\pi\mu}\int_{-\infty}^{\infty} x^{2k+1}\frac{d}{dx}e^{-\pi\mu|x|^2}dx \\
&= \frac{2k+1}{2\pi\mu}\int_{-\infty}^{\infty} x^{2k}e^{-\pi\mu|x|^2} \\
&= \mu^{-\frac{1}{2}-k}\frac{2k+1}{2(4\pi)^{k+1}k!}\partial_x^{2k}x^{2k} \\
&= \mu^{-\frac{1}{2}-k}\frac{1}{(4\pi)^{k+1}(k+1)!}\partial_x^{2(k+1)}x^{2(k+1)}
\end{aligned}$$

□

We consider integrals of the type

$$I(\tau) = \int_{\mathbb{R}^n} e^{\tau\phi(x)}a(x)dx$$

where a is smooth and compactly supported, ϕ is smooth with nonnegative real part and τ is supposed to be large. This integral is clearly bounded and we are interested in the behavior for large τ.

PROPOSITION 4.20. *Suppose that there exists $\kappa > 0$ so that*

$$|D\phi| - \operatorname{Re}\phi \geq \kappa > 0$$

in the support of ϕ *and that* $\operatorname{Re}\phi \le 0$. *Then, if* $N > 0$ *there exists* c_N *so that*

$$|I(\tau)| \le c_N \tau^{-N} \tag{4.18}$$

Proof. There exists a smooth function η so that

$$|\nabla \Phi| \ge \kappa/3$$

if $x \in \operatorname{supp}\eta \cap \operatorname{supp} a$ and

$$\operatorname{Re}\Phi < -\kappa/3$$

for $x \in \operatorname{supp}(1-\eta) \cap \operatorname{supp} a$. We split the two cases by writing

$$a = a\eta + a(1-\eta).$$

For the second term we get pointwise bounds of the integrand by $e^{-\tau\kappa/3}$, which gives the estimate for this term. So we may assume that

$$|D\phi| \ge \kappa$$

in the support of a. We calculate

$$\begin{aligned}\int e^{\tau\phi}(x)a(x)dx =& \tau^{-1}\int \frac{a\nabla\phi}{|\nabla\phi|^2}\nabla e^{\tau\phi}dx\\ =&\tau^{-1}\int e^{\tau\phi}\sum_j \partial_j \frac{a\partial_j\phi}{|\nabla\phi|^2}\,dx.\end{aligned}$$

where we used the divergence theorem to integrate by parts. We obtain the claim by induction. □

The next situation with a fairly complete understanding is the one dimensional situation.

Lemma 4.21 (Van der Corput). *We consider*

$$I = \int_c^d e^{ih(x)}a(x)dx$$

for real functions h *under the assumption that* a *is compactly supported and that for one* $j \ge 1$

$$h^{(j)} \ge \kappa > 0.$$

If $j = 1$ *we assume in addition that* h' *is monotone. Then*

$$|I| \le \kappa^{-\frac{1}{j}}(\|a'\|_{L^1} + 23^j\|a\|_{sup})$$

if a *is compactly supported.*

Proof. We begin with the case $j = 1$ and proceed by induction.

$$\begin{aligned}\int_c^d e^{ih(x)}a(x)dx =& i\int e^{ih}\partial_x(a/h')dx\\ =& i\int e^{ih}a_x/h'dx + i\int e^{ih}a\partial_x(1/h')dx\\ &+ ie^{ih(d)}a(d)/h'(d) - ie^{ih(c)}a(c)/h'(c)\end{aligned}$$

for the first term we obtain the estimate with L^1 and by monotony the second term is bounded by

$$\begin{aligned}\left|\int_c^d e^{ih} a\partial_x(1/h')dx\right| \leq & \|a\|_{sup}\left|\int_c^d \partial_x(1/h')dx\right| \\ = & \|a\|_{sup}|h'(c)^{-1} - h'(d)^{-1}| \\ \leq & \|a\|_{sup}\kappa^{-1}\end{aligned}$$

We assume that we have proven the estimate for $j-1 \geq 1$ and we want to prove it for j. So $h^{(j)} \geq \kappa$ and $h^{(j-1)}$ has at most one zero, which we assume to be 0. Let us assume that it is 0. We choose $\delta > 0$. If $|x| \geq \delta$ then $|h^{(j-1)}(x)| \geq \kappa\delta$. We apply the previous argument on both sides and obtain

$$|I| \leq 2\delta \sup_{|x|\leq\delta|a(x)|} + \delta^{-1/(j-1)}\kappa^{-1/(j-1)})(\|a'\|_{L^1} + 43^{j-1}\|a\|_{sup})$$

We choose $\delta = \kappa^{-1/j}$. This requires $c < -\delta$ and $d \geq \delta$, the modifications for the other cases are trivial. □

In higher dimensions the only large contributions can come from stationary points, i.e. points x in the support of a where $\operatorname{Re}\phi(x) = 0$ and $\nabla\phi(x) = 0$. We always assume that these points are nondegenerate in the sense that the Hesse matrix $D^2\phi$ is invertible. We assume that $\operatorname{Re}\phi \geq 0$.

THEOREM 4.22. *Under the assumptions above we assume that* 0 *is the only stationary point of* ϕ *in the support of* a *where* $\operatorname{Re}\phi(0) = 0$, *and that it is nondegenerate. Let*

$$I(\tau) = \int_{\mathbb{R}^n} e^{-\tau\pi\phi} a dx$$

We write

$$\phi = \phi(0) + x^T Ax + \psi(x).$$

Given N *there exists* M *and* C *so that*

(4.19)
$$\left| I - e^{-\tau\phi(0)}\tau^{-\frac{n}{2}-k}(\det A)^{-\frac12}\sum_{k=0}^M \frac{1}{(4\pi)^k k!}\left(\sum_{i,j=1}^n a_{ij}\partial_{ij}^2\right)^k (a(x)e^{-\tau\psi(x)})\Big|_{x=0}\right| \leq C\tau^{-N}.$$

There is a suggestive notation for the estimate:

$$I - e^{-\tau\pi\phi(0)}\tau^{-n/2}\det(A)^{-1/2}\exp\left(\frac{1}{4\pi\tau}\sum a_{ij}\partial_{ij}^2\right)(a(x)e^{-\tau\psi(x)})\Big|_{x=0} = O(\tau^{-\infty})$$

PROOF. Again it suffices to consider A_r positive definite. Let $\eta \in C_0^\infty(B_1(0))$, identically 1 on $B_{1/2}$. We may assume that a is supported in a small ball around $x = 0$ and that

$$\partial^\alpha\psi(0) = 0$$

for $|\alpha| \le 3$. We claim that for $\frac{1}{3} < s < \frac{1}{2}$

$$I_1(\tau) = \int (1 - \eta(x\tau^s)) e^{-\tau\pi\phi} a dx$$

and $N \in \mathbb{N}$ there exists C_N such that

$$|I_1(\tau)| \le c_N \tau^{-N}$$

We observe that

$$D|D\phi(x)|^{-1} \le c|x|^{-2}$$

near $x = 0$ in the support of a and each integration by parts brings a gain of τ^{-1}. After many integrations by parts we end up with

$$C\tau^{-k} \int_{2|x| \ge \tau^{-s}} |x|^{-2k} dx \le C\tau^{-k}\tau^{2ks}.$$

Since $s < \frac{1}{2}$ this implies the assertion. Let p_M be the Taylor expansion of $ae^{-\tau\pi\psi}$ at 0 of order M. Then

$$|p_M - ae^{-\tau\pi\psi}| \le C_M |x|^{M+1} \tau^{\frac{M+1}{3}}$$

$$|I_2(\tau)| = \left| \int \eta(x\tau^{1/4}) e^{-\pi\tau x^T A x} (p_M - ae^{-\tau\pi\psi}) dx \right| \le c\tau^{(M+1)(\frac{1}{3}-s)}$$

We obtain the statement by chosen M large. Finally we consider - under the assuming that $\operatorname{Re} A$ is positive definite -

$$I_3(\tau) = \int (1 - \eta(x\tau^s)) e^{-\tau\pi x^T A x} p_M dx$$

in the same fashion as the first term, using also the Gaussian decay, which, however, does not enter in quantitative fashion since the integration by parts leads to factors $|x|^{-N}$, which ensures integrability at ∞. This shows that

$$\left| I(\tau) - \int (1 - \eta(x\tau^s)) e^{-\tau\pi\phi(0)} e^{-\tau\pi x^T A x} p_M dx \right| \le c_N \tau^{-N}$$

if M is sufficiently large. The second term has been evaluated in Proposition 4.19.

□

4.1. Example 1: Korteweg-de Vries equation and the Airy function. The Korteweg-de Vries

$$u_t + u_{xxx} + 6uu_x = 0$$

is among the most fascinating partial differential equation. It describes one dimensional water waves, for which it describes a second order approximation: To first order there is a linear transport resp. wave equation, but waves are in general slower than the wave speed of the wave equation.

It it strongly related to Schrödinger operators

$$\psi \to \psi_{xx} + u\psi.$$

There is a complicated but amazingly explicite way of getting formulas for solutions via the so called inverse scattering theory.

Here we will look at the linear part

$$u_t + u_{xxx} = 0$$

By a Fourier transform

$$\hat{u}(\xi) = e^{it(2\pi)^3\xi^3}\hat{u}_0(\xi)$$

and with

$$g(x) = \int e^{i[(2\pi)^3\xi^3+2\pi x\xi]}d\xi$$

$$u(t) = t^{-1/3}g(.t^{-1/3}) * \hat{u}_0$$

The Airy function is defined by

$$\mathrm{Ai}(x) = (2\pi)^{-1}\int e^{i(\xi^3/3+ix\xi)}d\xi$$

which, as usual, is an abuse of notation and has to be as limit of the integral with the additional factor $e^{-\varepsilon\xi^2}$ or as inverse Fourier transform. Then

$$g(x) = 3^{1/3}(2\pi)^{-2}\,\mathrm{Ai}(x(3t)^{-1/3}).$$

By the Lemma of van der Corput $\mathrm{Ai}(x)$ is uniformly bounded, and since we can differentiate under the integral and apply Proposition 4.20 it is a smooth function (the noncompact interval does not change this fact, but one has to pay attention to what happens at infinity). The phase function is $\phi(\xi) = \xi^3/3 + x\xi$ with derivative $\xi^2 + x$.

We consider first the case when x is negative and there are two stationary points $\xi = \pm\sqrt{-x}$. Let $x < -10$ and choose a cutoff function $\eta \in C_0^\infty((-2,2))$, identically 1 in $(-1,1)$ and write

$$\begin{aligned}\mathrm{Ai}(x) =&(2\pi)^{-1}\int \eta(\xi-\sqrt{-x})\int e^{i(\xi^3/3+ix\xi)}d\xi\\ &+(2\pi)^{-1}\int \eta(\xi+\sqrt{-x})\int e^{i(\xi^3/3+ix\xi)}d\xi\\ &+(2\pi)^{-1}\int \left[1-\eta(\xi+\sqrt{-x})-\eta(\xi-\sqrt{-x})\right]\int e^{i(\xi^3/3+ix\xi)}d\xi\end{aligned}$$

The last term decays faster than any negative power of $|x|$ by Proposition 4.20. The second is the complex conjugate of the first one. We apply Theorem 4.22 to the first term and obtain the first part of

Lemma 4.23 (Asymptotics of Airy function).

$$\mathrm{Ai}(x) = \pi^{-1/2}(-x)^{-\frac14}\sin(\frac23(-x)^{3/2}+\frac{\pi}{4}) + O(r^{-\frac12}$$

if $x < 0$ and

$$\left|\mathrm{Ai}(x) - (2\pi)^{-1}x^{-\frac14}\exp(-\frac23 x^{3/2})\sum_{k=0}^{N}(-9)^k\Gamma(3k+\frac12)/(2k)!x^{-\frac{3k}{2}}\right| \le c_N\exp(-\frac23 x^{3/2})x^{-\frac14-\frac{3k}{2}}.$$

It remains to prove the asymptotics for $x > 0$. We first verify for $x > 1$

$$\mathrm{Ai}(x) = (2\pi)^{-1}\int e^{i(\xi+i\eta)^3+ix(\xi+i\eta)}d\xi$$

for $0 \le \eta \le \xi$, by differentiating with respect to η, and checking continuity at $\eta = 0$. Then

$$\begin{aligned} \mathrm{Ai}(x) =& (2\pi)^{-1} \int e^{i(\xi+i\sqrt{x})^3/3 + ix(\xi+i\sqrt{x})} d\xi \\ =& (2\pi)^{-1} e^{-\frac{2}{3}x^{3/2}} \int e^{-\sqrt{x}\xi^2 + i\xi^3/3} d\xi. \end{aligned}$$

The proof of Proposition 4.22 consists of taking a Taylor expansion. Here we take a Taylor expansion of $\exp i\xi^3/3$ and evaluate the integrals.

Mutliplication by $|\xi|^s$ on the Fourier side roughly corresponds to taking s derivatives. Looking at the expansion of the Airy function suggests that about half a derivative of the Airy function is bounded. A precise version of that is the contents of the next lemma.

PROPOSITION 4.24. *There exists C so that*

$$\left| \int |\xi|^{\frac{1}{2}+is} e^{i(\xi^3/3 + x\xi)} d\xi \right| \le C(1+|s|)$$

for $s \in \mathbb{R}$.

PROOF. We decompose the integral into an integral over $(-2, 2)$ which is always bounded, and integrals over $(-\infty, -1)$ and $(1, \infty)$ using η as above. Then estimate follows if we prove the estimate

$$\left| \int_0^\infty (1-\eta(\xi)) |\xi|^{\frac{1}{2}+is} e^{i(\xi^3/3+x\xi)} d\xi \right| \le C(1+|s|).$$

If $x > -1/2$ we define $t = \xi^3/3 + x\xi$ and get

$$\begin{aligned} &\int_0^\infty (1-\eta(\xi)) |\xi|^{\frac{1}{2}+is} e^{i(\xi^3/3+x\xi)} d\xi \\ &\qquad = \int_0^\infty (1-\eta(\xi(t))) |\xi(t)|^{\frac{1}{2}+is} \frac{1}{(\xi(t)^2+x)^2} \frac{d}{d\xi} e^{i(\xi^3/3+x\xi)} \end{aligned}$$

we integrate once by parts to obtain the desired estimate for $x \ge -1/2$.

If $x < 0$ we decompose into $[\pm\sqrt{-x} - 2, \pm\sqrt{x} + 2]$ and the complement. On the complement we argue as for $x > -1/2$. In these intervals we apply the Lemma of von der Corput with $j = 2$. The second derivative is bounded from below by $(-x)^{1/2}$ and

$$\frac{d}{d\xi} |\xi|^{1/2+is} = (\frac{1}{2} + is) \frac{\xi}{|\xi|} |\xi|^{-\frac{1}{2}+is}$$

which gives the estimate. □

LEMMA 4.25. *Let $S(t)$ be the unitary group defined by the Airy equation. Then*

$$\| |D|^{\frac{1}{p}-\frac{1}{2}} S(t) u_0 \|_{L^{p'}(\mathbb{R})} \le c|t|^{-(\frac{1}{p}-\frac{1}{2})} \|u_0\|_{L^p(\mathbb{R})}$$

Here $|D|^s u = \mathcal{F}^{-1}(|\xi|^s \hat{u})$. The analogue of the Strichartz estimate is an immediate consequence.

PROOF. Let for $0 \le \operatorname{Re} z \le \frac{1}{2}$

$$T_z u_0 = e^{z^2} \mathcal{F}^{-1}(e^{i\xi^3/3} \hat{u}_0)$$

If $z = is$ then by Plancherel

$$\|T_{is}u_0\|_{L^\infty} \le \|u_0\|_{L^2}$$

and if $z = \frac{1}{2} + is$ $T_{\frac{1}{2}+is}$ is the convolution by a bounded function according to the previous Proposition by

$$ce^{\frac{1}{4}-s^2}(1+|s|)$$

which is uniformly bounded, hence the operator is bounded from L^1 to L^∞. By the Theorem of Riesz-Thorin on complex interpolation

$$\|T_t u_0\|_{L^{p'}} \le c\|u_0\|_{L^p}$$

if $1 \le p \le 2$, $0 \le t \le \frac{1}{2}$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{p'} = t$. This implies the assertion for $t = 1$. For general t

$$g_t(t,x) = t^{-1/3} g_1(x/t^{1/3})$$

and

$$\||D|^{\frac{1}{2}} g_t\|_{L^\infty} \le c|t|^{-1/2}.$$

The same argument yields the claimed estimate. □

4.2. Example 2: The half wave equation.

5. Quantization

Quantization means a map from functions on $\mathbb{R}^n \times \mathbb{R}^n$ to operators on $\mathbb{R}^n$. We choose a so called semiclassical calculus with a semi classical parameter $h > 0$, basically Planck's constant in physics.

We begin with several preliminary observations. Denote $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. If $u \in \mathcal{S}(\mathbb{R}^n)$ and $v \in \mathcal{S}(\mathbb{R}^m)$ then $(x, y) \to u(x)v(y) \in \mathcal{S}(\mathbb{R}^{n+m})$. If $T \in \mathcal{S}'(\mathbb{R}^m)$ and $u \in \mathcal{S}(\mathbb{R}^{n+m})$ then $x \to T(u(x, .)) \in \mathcal{S}(\mathbb{R}^n)$ and for the partial Fourier transform we have

$$(\mathcal{F}_y u)(x, \eta) \in \mathcal{S}(\mathbb{R}^{n+m}).$$

5.1. Semiclassical Fourier transform.

We define

$$\mathcal{F}_h\phi(\xi) = \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle x,\xi\rangle}\phi(x)dx = \hat{\phi}(\xi/(2\pi h))$$

with inverse

$$\mathcal{F}_h^{-1}\phi(x) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle x,\xi\rangle}\phi(\xi)d\xi.$$

Our Fourier transform is the one with $h = \frac{1}{2\pi}$ and we could restrict to this h in the sequel.

We denote point in $\mathbb{R}^n \times \mathbb{R}^n$ by (x, ξ) and we call x position and ξ momentum.

5.2. Definition of quantization.

DEFINITION 4.26. *Let $a \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. We define the Weyl quantization to be the operator*

$$(a^w(x,hD)u)(x) := \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(\frac{x+y}{2},\xi)u(y)dyd\xi, \tag{4.20}$$

the standard quantization

$$(a(x,hD)u)(x) := \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi)u(y)dyd\xi, \tag{4.21}$$

and, for $0 \le t \le 1$, the t quantization

$$(Op_t(a)u)(x) := \frac{1}{2\pi h}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(tx+(1-t)y,\xi)u(y)dyd\xi. \tag{4.22}$$

We denote by

$$D^j = \frac{1}{i}\partial_{x_j}$$

and D^α for multiindices α with the obvious meaning.

(1) $Op_t(\xi^\alpha) = (hD)^\alpha u$
(2) If $a(x,\xi) = \sum_{|\alpha|\le N} a_\alpha(x)\xi^\alpha$ then

$$a(x,hD)u = \sum_{|\alpha|\le N} a_\alpha(x)(hD)^\alpha u$$

(3) $\langle x, hD\rangle^w u = \frac{h}{2}\langle D, xu\rangle + \frac{h}{2}\langle x, Du\rangle$

LEMMA 4.27. *Let $a \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ and $0 \le t \le 1$. Then $Op_t(a)$ defines a continuous linear operator from $\mathcal{S}'$ to $\mathcal{S}$. The formal adjoint is $Op_t(a)^* = Op_{1-t}(\bar{a})$.*

PROOF. We have

$$\mathrm{Op}_t(a)u(x) = \int_{\mathbb{R}^n} K_t(x,y)u(y)dx$$

with

$$\begin{aligned} K_t(x,y) =& \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(tx+(1-t)y,\xi)d\xi \\ =& \left[\mathcal{F}_h^{-1}(a(tx+(1-t)y,.)\right](x-y) \end{aligned}$$

The partial Fourier transform $g(z,w) = (\mathcal{F}^{-1}a(z,.))(w)$ is a Schwartz function by the considerations above. The equations

$$tx + (1-t)y = z, x - y = w$$

define linear maps, and hence K_t is a Schwartz function. Let $T \in \mathcal{S}'(\mathbb{R}^n)$. Then

$$x \to T(K_t(x,.)) \in \mathcal{S}(\mathbb{R}^n)$$

which proves the first assertion.

The statement about the formal adjoint follows from the definition. □

In particular, if a is real then $a^w(x,hD)$ is formally self adjoint.

LEMMA 4.28. *Let $a \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ and $0 \le t \le 1$. Then $Op_t(a)$ defines a continuous linear operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.*

PROOF. By the same argument as above the partial inverse Fourier transform defines a distribution, and the contructions above of the kernel defines a tempered distribution T on $\mathbb{R}^n \times \mathbb{R}^n$. If $u, v \in \mathcal{S}(\mathbb{R}^n)$ then

$$u(x)v(y) \in \mathcal{S}(\mathbb{R}^n).$$

and, given $v \in \mathcal{S}(\mathbb{R}^n)$ the map $u \to T(u(x)v(y))$ defines a unique tempered distribution which coincides with the quantization whenever the previous definition applies. □

5.3. Simple examples.

LEMMA 4.29. *If $a(x,\xi) = b(x)$ then*

$$Op_t(a)u = b(x)u$$

PROOF. The claim follows from Fourier inversion if $t = 0$. It suffices to proof it for $a \in \mathcal{S}(\mathbb{R}^n)$. Let $u \in \mathcal{S}$ and compute

$$\begin{aligned}
&\partial_t \operatorname{Op}_t u \\
&= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} \sum_{j=1}^n (\partial_j b)(tx + (1-t)y)(x_j - y_j) u(y) dy d\xi \\
&= \frac{1}{ih(2\pi h)^n} \int_{\mathbb{R}^n} \sum_{j=1}^n \partial_{\xi_j} \cdot \left(\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} (\partial_j b)(tx + (1-t)y) u(y) dy \right) d\xi \\
&= 0
\end{aligned}$$

by the divergence theorem (applied on balls with radii tending to infinity, and using the decay of the Schwartz functions), and since the inner integral defines a Schwartz function. □

LEMMA 4.30. *Let $l(x,\xi) = \langle x^*, x\rangle + \langle \xi^*, \xi\rangle$ be a linear function ($x^*, \xi^* \in \mathbb{R}^n$). Then*

$$Op_t(l)u = \langle x^*, x\rangle u + \langle \xi^*, hDu\rangle.$$

If $c(x,\xi) = \sum_{J=1}^n c_j(x)\xi_j$ then

$$c^w(x, hD)u = \frac{h}{2} \sum_{j=1}^n (D^j(c_j u) + c_j D^j u)$$

PROOF. The first statement is a consequence of the previous considerations. For the second we use the definition

$$\begin{aligned}
c^w(x, hD)u =& \frac{1}{(2\pi h)^n} \sum \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} c_j(\frac{x+y}{2}) \xi_j e^{\frac{i}{h}\langle x-y,\xi\rangle} u(y) d\xi dy \\
=& -\frac{1}{(2\pi h)^n} \sum \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} c_j(\frac{x+y}{2}) h D_{y_j} e^{\frac{i}{h}\langle x-y,\xi\rangle} u(y) d\xi dy \\
=& \frac{1}{(2\pi h)^n} \sum \frac{h}{i} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_{x_j} c_j(\frac{x+y}{2}) e^{\frac{i}{h}\langle x-y,\xi\rangle} u(y) d\xi dy \\
&+ \frac{1}{(2\pi h)^n} \sum \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} c_j(\frac{x+y}{2}) e^{\frac{i}{h}\langle x-y,\xi\rangle} h D_{y_j} u(y) d\xi dy \\
=& \frac{h}{2i} (\partial_{x_j} c_j) u + h \sum c_j(x) D_{x_j} u
\end{aligned}$$

This implies the second statement. □

LEMMA 4.31. *The following identities hold:*

$$(D_{x_j}a)^w = [D_{x_j}, a^w]$$

and

$$h(D_{\xi_j}a)^w = -[x_j, a^w]$$

There is symplectic quadratic form

$$\sigma(z,w) = \xi\cdot y - x\cdot\eta, \quad z=(x,\xi), w=(y,\eta) \tag{4.23}$$

on $\mathbb{R}^{2n}$. We can write it via the Euclidean inner product

$$\sigma(z,w) = \langle Jz, w\rangle \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

LEMMA 4.32.

$$\left(e^{\frac{i}{h}l}\right)^w u = e^{\frac{i}{h}l(x,D)}u := e^{\frac{i}{h}\langle x^*,x\rangle + \frac{i}{2h}\langle x^*,\xi^*\rangle}u(x+\xi^*).$$

If $l, m \in \mathbb{R}^{2n}$ *then*

$$e^{\frac{i}{h}l(x,hD)}e^{\frac{i}{h}m(x,hD)}u = e^{\frac{i}{2h}\sigma(l,m)}e^{\frac{i}{h}(l+m)(x,hD)}$$

Consider the PDE

$$ih\partial_s v + l(x,hD)v = 0$$

Its solution is - independent of t

$$v(1,x) = e^{\frac{i}{h}\langle x^*,x\rangle + \frac{i}{2h}\langle x^*,\xi^*\rangle}u(x+\xi^*).$$

which is the meaning of the not rigorously defined second equality.

PROOF.

$$\begin{aligned}
(e^{\frac{i}{h}l})^w u =& \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}e^{\frac{i}{h}(\langle\xi^*,\xi\rangle + \langle x^*,\frac{x+y}{2}\rangle)}u(y)dyd\xi \\
=& \frac{e^{\frac{i}{h}\langle x^*,x\rangle}}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y+\xi^*,\xi\rangle}(e^{\frac{i}{2h}\langle x^*,y\rangle}u(y))dyd\xi \\
=& \frac{e^{\frac{i}{h}\langle x^*,x\rangle}}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle}(e^{\frac{i}{2h}(\langle x^*,y+\xi^*\rangle)}u(y+\xi^*))\,dy\,d\xi \\
=& e^{\frac{i}{h}\langle x^*,x\rangle + \frac{i}{2h}\langle x^*,\xi^*\rangle}u(x+\xi^*).
\end{aligned}$$

This implies the first assertion. The second assertion follows from an application of the first formula: Let $l = (x^*,\xi^*)$ and $m = (y^+;\eta*)$. Then

$$\begin{aligned}
(e^{\frac{i}{h}l})^w(e^{\frac{i}{h}})^w u(x) =& (e^{\frac{i}{h}l})^w\left[e^{\frac{i}{2h}\langle y*,\xi^*\rangle}e^{\frac{i}{h}\langle y^*,.\rangle}u(.+\eta^*)\right] \\
=& e^{\frac{i}{2h}(\langle x^*,\xi^*\rangle + \langle y^*,\eta^*\rangle)}e^{\frac{i}{h}\langle y^*,\xi^*\rangle}e^{\frac{i}{h}\langle x^*+y^*,x\rangle}u(x+\eta^*+\xi^*) \\
=& e^{\frac{i}{2h}(\langle y^*,\xi^*\rangle - \langle x^*,\eta^*\rangle)}(e^{\frac{i}{h}(l+m)})^w u(x).
\end{aligned}$$

□

If symbols only depend on ξ and not on x then all quantization yield the same result.

5.4. Composition of semiclassical pseudodifferential operators.

THEOREM 4.33. *Let Q be a nonsingular selfadjoint $n \times n$ matrix. Then*

$$e^{\frac{ih}{2}\langle QD,D\rangle}u(x) = \frac{|\det Q|^{-\frac{1}{2}}}{(2\pi h)^{n/2}} e^{\frac{i\pi}{4}\operatorname{sign} Q} \int_{\mathbb{R}^n} e^{-\frac{i}{2h}\langle Q^{-1}y,y\rangle} u(x+y)dy$$

$$e^{ih\sigma(D_z,D_w)}u(z,w) = \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} e^{-\frac{i}{h}\sigma(z_1,w_1)} u(z+z_1, w+w_1)dz_1dw_1$$

Here $\operatorname{sign} Q$ is the signature, the number of positive eigenvalues minus the number of negative eigenvalues.

PROOF. The second statement is a special case of the first one with the quadratic form given by the matrix

$$\begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}.$$

We compute

$$\begin{aligned} e^{\frac{i}{h}\langle QD,D\rangle}u(x) =& e^{\frac{i}{2h}\langle QhD,hD\rangle}u(x) \\ =& \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} e^{\frac{i}{2h}\langle Q\xi,\xi\rangle} u(y)dyd\xi \\ =& \frac{|\det Q|^{-\frac{1}{2}}}{(2\pi h)^{n/2}} e^{\frac{i\pi}{4}\operatorname{sign} Q}\int e^{-\frac{i}{2h}\langle Q^{-1}(x-y),x-y\rangle}u(y)dy \\ =& \frac{|\det Q|^{-\frac{1}{2}}}{(2\pi h)^{n/2}} e^{\frac{i\pi}{4}\operatorname{sign} Q}\int e^{-\frac{i}{2h}\langle Q^{-1}(y),y\rangle}u(x+y)dy \end{aligned}$$

where we exchanged the order of integration (first multiplying by $e^{-\varepsilon|\xi|^2}$ and then we passed to the limit as $\varepsilon \to 0$. The ξ integration yields the inverse Fourier transform of the complex Gaussian. The last integral follows by the translation invariance of the integration. □

With Proposition 4.22 we get an asymptotic series for this integral in powers of h.

THEOREM 4.34.

$$\mathcal{F}^{-1}a^w(x,hD)\mathcal{F} = a^w(hD,-x)$$

PROOF. We prove the claim for $a \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. The integral kernel $K_h(x,y)$ of $\mathcal{F}_h^{-1}a^w\mathcal{F}_h$ is

$$\begin{aligned} &\frac{1}{(2\pi h)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}(\langle x',x\rangle+\langle x'-y',\zeta\rangle-\langle y',y\rangle)}a((x'+y')/2,\zeta)dy'dx'd\zeta \\ &= \frac{1}{(2\pi h)^{2n}}2^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}2\left(\langle x',\zeta+\frac{x+y}{2}\rangle-\langle z,y+\zeta\rangle\right)}a(z,\zeta)dx'dzd\zeta \\ &= \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,z\rangle}a(z,-\frac{x+y}{2})dz \end{aligned}$$

where we used in the last step that

$$\mathcal{F}_h^{-1}(e^{\frac{2i}{h}\langle x',\zeta+\frac{x+y}{2}\rangle}) = 2^n\delta_0(\zeta+\frac{x+y}{2}).$$

□

Interpretation: The map $(x, \xi) = J(x, \xi)$ is a symplectic map. The semiclassical Fourier transform quantizes the symplectic map J.

We define

$$A(D) = \sigma((D_x, D_\xi); (D_y, D_\eta))$$

THEOREM 4.35 (Composition for Weyl quantization). *With*

$$a\#b(x,\xi) := e^{ihA(D)}(a(x,\xi)b(y,\eta))\Big|_{y=x,\xi=\eta}$$

we have

$$(a\#b)^w u = a^w b^w u$$

and with

$$c(x,\xi) = e^{\frac{i}{h}\sigma(h(D_x,D_\xi,D_y,D\eta)}(a(x,\xi)b(y,\eta)\Big|_{x=y,\xi=\eta}$$

$$a(x, hD)b(x, hD)u = c(x, hD)u.$$

PROOF. Let

$$\hat{a}(l) = \int_{\mathbb{R}^{2n}} e^{-\frac{i}{h}\langle l,(x,\xi)\rangle} a(x,\xi)dxd\xi$$

By the inversion formula

$$a^w(x, hD) = \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^{2n}} \hat{a}(l)e^{\frac{i}{h}l(x,hD)}dl$$

We apply this formula for a and b to get

$$\begin{aligned} a^w(x,hD)b^w(x,hD) =& \frac{1}{(2\pi h)^{4n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} \hat{a}(l)\hat{b}(m)e^{\frac{i}{h}l(x,hD)}e^{\frac{i}{h}m(x,hD)}dmdl \\ =& \frac{1}{(2\pi h)^{2n}}\int_{\mathbb{R}^{2n}} \hat{c}(r)e^{\frac{i}{h}r(x,hD)}dr \end{aligned}$$

where, by Lemma 4.32

$$\begin{aligned} \hat{c}(r) =& \frac{1}{(2\pi h)^{2n}}\int_{\mathbb{R}^{2n}} \hat{a}(l)\hat{b}(r-l)e^{\frac{i\sigma(l,r-l)}{2h}}dl \\ =& \frac{1}{(2\pi h)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}\left(\frac{1}{(2\pi h)^{2n}}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle l+m-r,z\rangle}dz\right)e^{\frac{i}{2h}\sigma(l,m)}\hat{a}(l)\hat{b}(m)dldm \end{aligned}$$

Hence, since

$$e^{\frac{i}{2h}\sigma(hD_z,hD_w)}e^{\frac{i}{h}(\langle l,z\rangle+\langle m,w\rangle)} = e^{\frac{i}{2h}\sigma(l,m)}e^{\frac{i}{h}(\langle l,z\rangle+\langle m,w\rangle)}$$

$$c(z) = \frac{1}{(2\pi h)^{4n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} e^{\frac{i}{2h}\sigma(hD_z,hD_w)}e^{\frac{i}{h}(\langle l,z\rangle+\langle m,w\rangle)}\Big|_{z=w}\hat{a}(l)\hat{b}(m)dldm$$

The statement follows now by Theorem 4.33.

The functions are called *symbols*. The proposition allows to compute the symbol of the composition from the symbols of the operators. The method of stationary phase allows to obtain an expansion in terms of powers of h. This expansion is given by formally expanding the exponential, and then applying the operators to the functions.

Since there is exactly one stationary point we can apply this expansion even if we have unbounded symbols. The first term is always the product. The second term is

$$A(D)a(x,\xi)b(y,\eta)|_{x=y,\xi=\eta} = h\sum_{j=1}^n \partial_{x_j} a \partial_{\xi_j} b - \partial_{\xi_j} a \partial_{x_j} b.$$

This is the first term in the expansion of the operator. The expression on the RHS is h times the Poisson bracket of the symbols a and b, $h\{a,b\}$.

Here we already see an important property of the semiclassical limit $h \to 0$: The composition becomes closer and closer to a multiplication.

This suggest that as first approximation to composition of operators we may consider the multiplication. In particular we expect some approximate inverse to be given by taking the inverse of the symbol. So if we consider the elliptic equation with smooth coefficients

$$-a^{ij}\partial_j^2 u = f$$

then the operator is given by the classical symbol $a^{ij}(x)\xi_i\xi_j$. One can proof that

$$D[(1+a^{ij}(x)\xi_i\xi_j)^{-1,w}a^{ij}\partial_{ij}^2 - 1]$$

is bounded operator on L^p.

To carry that program out we need

(1) Criteria for L^p boundedness
(2) A calculus for pseudodifferential operators, i.e. the operators obtained by quantizing symbols.

□

6. Cotlar's Lemma and the Theorem of Calderón-Vaillancourt

In this section we will prove L^2 boundedness of semiclassical operators under fairly weak assumptions on the symbol a. As a start we consider Schwartz functions.

LEMMA 4.36. *Let $a \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$. Then*

$$\|\operatorname{Op}_t(a)f\|_{L^2(\mathbb{R}^n)} \le c(n) \sup_{|\alpha|,|\beta|\le n+1} \sup_{x.\xi} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \|f\|_{L^2}$$

It suffices to assume that the quantity on the right hand side is bounded.

PROOF. The partial Fourier transform $\check a = \mathcal{F}_h^{-1}{}_z a(w,.)(z)$ is a Schwartz function. It satisfies

$$|z^\beta \check a| = |\mathcal{F}^{-1}((hD)_\xi^\beta a)| \le c(n) \sup_{|\gamma|\le n+1} h^{-|\beta|}|\xi^\gamma \partial_\xi^\beta a(x,\xi)|$$

We obtain for the integral kernel

$$K_t(x,y) = h^{-n}\check a(tx+(1-t)y,(x-y))$$

and

$$|K_t(x,y)| \le c(n)h^{-n}(1+|x-y|/h)^{-n-1} \sup_{|\alpha|,|\beta|\le n+1} \sup_{w,\xi} |\xi^\alpha \partial_\xi^\beta a|.$$

Then

$$\sup_x \int |K_t(x,y)|dy + \sup_y \int |K_t(x,y)|dx \le c_2(n) \sup_{|\alpha|,|\beta|\le n+1} \sup_{w,\xi} |\xi^\alpha \partial_\xi^\beta a|$$

and the assertion follows from Schur's lemma 2.17. □

The product of two functions with disjoint support is zero. This fails to be true for the composition of pseudodifferential operators, but it remains approximately true. We focus on $h=1$ in the sequel and recover the assertion for $0<h$ by a standard rescaling.

LEMMA 4.37. *Let $h>0$, $\tilde u(x) = u(h^{1/2}x)$ and $a_h(x,\xi) = a(h^{\frac12}x, h^{\frac12}\xi)$. Then*

$$a^w(h^{\frac12}x, hD)u(h^{\frac12}x) = a_h^w(x,D)\tilde u.$$

PROOF.

$$\begin{aligned} a^w(h^{1/2}x, hD)&u(h^{1/2}x) \\ &= \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a\Big(\frac{h^{1/2}x+y}{2},\xi\Big)e^{\frac{i}{h}\langle h^{1/2}x-y,\xi\rangle}u(y)dyd\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a_h\Big(\frac{x+y}{2},\xi\Big)e^{i\langle x-y,\xi\rangle}\tilde u(y)dyd\xi \end{aligned}$$

□

LEMMA 4.38. *Let $a,b\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ with*

$$d = d(\operatorname{supp} a, \operatorname{supp} b) > 0$$

and

$$|\partial_x^\alpha \partial_\xi^\beta a| + |\partial_x^\alpha \partial_\xi^\beta a| \le B \qquad \text{for } x,\xi\in\mathbb{R}^n, |\alpha|,|\beta|\le 6n+N+2$$

Then

$$\|a^w(x,D)b^w(x,D)f\|_{L^2} \le c(n,N)d^{-N}B^2\|f\|_{L^2}.$$

PROOF. We recall that the symbol of the composition is

$$c(z) = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} e^{-\frac{i}{2}\sigma(w_1,w_2)}a(z+w_1)b(z+w_2)dw_1dw_2$$

The point 0 is the unique critical point, and the integrand vanishes for $|(w_1,w_2)|\le d/\sqrt2$. The integrations by parts in the stationary phase argument lead to bound (since the same type of argument holds for derivatives)

$$|\partial^\alpha c| \le c_N B^2 d^{-N}.$$

This implies the statement of the Theorem of Calderón-Vaillancourt below. At this point we prove the statement under the additional assumption that b is supported in a unit ball $B_1(x_0,\xi_0)$ Then

$$|\partial_z^\alpha c(z)| \le c d^{-N}(1+|z-(x_0,\xi_0)|)^{-M}.$$

This implies that the symbol c is a shifted Schwartz function. Since

$$[c^w(x,D)u](x+x_0) = c^w(x+x_0,D)u(.+x_0)$$

and

$$e^{-i\langle\xi_0,x\rangle}c^w(x,D)(e^{i\langle\xi_0,y\rangle}u(y))(x) = c^w(x,D-\xi_0)u$$

and neither translation nor multiplication by the complex exponential changes the L^2 the boundedness assertion follows from Lemma 4.36.

It remain to count the number of derivatives we need. For the application of (4.36) we need $n+1$ derivatives and a decay of power $n+1$. Each derivative in the argument above gives one inverse power of d resp. one inverse power of $|z-(x_0,\xi_0)|$ or one of $|w|^{-1}$. The integration is over a $4n$ dimensional space and it uses up $4n$ inverse powers of w. Thus we need $4n+(2n+2)+N$ derivatives for this argument. It is clearly not optimal.

This special case will suffice to proof the Theorem of Calderón Vaillancourt 4.40, which in turn implies the full statement of the proposition here. □

LEMMA 4.39 (Cotlar's lemma). *Let H be a Hilbert space and $T_j : H \to H$ a family of operators, and $\gamma : \mathbb{Z} \to \mathbb{R}^+$ such that*

$$\|T_j^* T_k\|_{L(H)} \le \gamma^2(j-k), \|T_j T_k^*\|_{L(H)} \le \gamma^2(j-k)$$

for all j,k. Suppose that

$$\sum_{l=-\infty}^{\infty} \gamma(l) =: A < \infty$$

Then

$$\|\sum_{j=1}^N T_j\|_{L(H)} \le A$$

for all N.

PROOF. We have with $T = \sum_{j=1}^N T_j$

$$(T^*T)^n = \sum_{j_1,\dots j_n,k_1,\dots,k_n=1}^N T_{j_1}^* T_{k_1} \dots T_{j_n}^* T_{k_n}$$

Now

$$\|T_{j_1}^* T_{k_1} \dots T_{j_n}^* T_{k_n}\| \le \|T_{j_1}\| \|T_{k_n}\| \prod \|T_{k_i} T_{j_{i+1}}^*\|$$

$$\|T_{j_1}^* T_{k_1} \dots T_{j_n}^* T_{k_n}\| \le \prod \|T_{j_i}^+ T_{k_i}^*\|$$

We multiply and take square roots

$$\|T_{j_1}^* T_{k_1} \dots T_{j_n}^* T_{k_n}\| \le (\|T_{j_1}\| \|T_{k_n})^{1/2} \| \prod \|T_{k_i} T_{j_{i+1}}^*\|^{1/2} \prod \|T_{j_i}^+ T_{k_i}^*\|^{1/2}$$

and thus with $B = \sup \|T_j\|$

$$\|(T^*T)^n\| \le \sum_{j_1,\dots j_n,k_1,\dots,k_n=1}^N B \prod \gamma(j_i - k_i)\gamma(k_i - j_{i+1}) \le NBA^{2n-1}$$

Since T^*T is selfadjoint

$$\|T\|_{L(H)}^2 = \|T^*T\|_{L(H)} = \|(T^*T)^n\|_{L(H)}^{1/n} \le (NB)^{1/n} A^{2-\frac{1}{n}}$$

We let n tend to infinity to obtain the assertion. □

Let $a \in C^{8n+4}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfy

$$\sup_{x,\xi,|\alpha|+|\beta| \le c(n)} |x^\beta \partial_x^\alpha a(x,\xi)| \le B \tag{4.24}$$

LEMMA 4.40 (Calderón-Vaillancourt). *, The operators*

$$T = a^w(x, D)$$

satisfies

$$\|Tf\|_{L^2} \le c(n)B\|f\|_{L^2}.$$

PROOF. We choose a function η supported in $[-1,1]^n$ so that

$$\sum_{k\in\mathbb{Z}^n} \eta(x-k) = 1.$$

(Choose $\tilde\eta \in C_0^\infty((-1,1)^{2n})$ nonnegative and identially 1 on $[-1/2,1/2]^{2n}$. Then set

$$\eta(z) = \frac{\tilde\eta(z)}{\sum_k \tilde\eta(z+k)}$$

Given $k \in \mathbb{Z}^{2n}$ we define $a_k(z) = a(z)\eta(z+k)$

$$T_k f = a_k^w f$$

We claim that

$$\|T_k^* T_l\|_{L(L^2)} + \|T_k^* T_l\|_{L(L^2)} \le C(1+|k-l|)^{-2n-1} \tag{4.25}$$

Since the square roots of the bounds on the RHS are summable this implies

$$\|\sum_{|k|\le N} T_{kl} f\|_{L^2} \le C\|f\|_{L^2}.$$

It is not difficult to let N tend to infinity.

Both terms on the left hand side have the same structure and it suffices to deal with one of them. Now

$$T_k^* T_l = \overline{a_k}^w(x,D) a_l^w(x,D)$$

We apply Proposition 4.36 twice to see that

$$\|T_k^* T_l\|_{L(L^2)} \le CB^2$$

and, if $|k-l| \ge 2n$, by Proposition 4.38

$$\|T_k^* T_l\|_{L(L^2)} \le cB^2|k-l|^{-2n-1}.$$

□

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CHAPTER 5

Singular integrals of Calderón-Zygmund type

1. The setting

The basic setting of this chaper are spaces of homogeneous type (X, d, μ) where (X, d) is a complete metric space and μ is a Borel measure, finite on bounded sets, which satisfies a doubling condition: There exists b so that

$$\mu(B_{3r}(x)) \le b\mu(B_r(x))$$

LEMMA 5.1. *Let A be a closed bounded set. Then A is compact.*

PROOF. We prove that A is totally bounded. Closed totally bounded sets in a complete metric space are compact.

Since A is bounded there exists a ball $A \subset B_R(x_0)$.

Let $R > r > 0$. There is at most a finite number of disjoint balls with centers in A with radii $= r/3$ by the doubling condition: there exists k so that

$$R \le 3^{k-3} r.$$

Let $B_{r/3}(x_j)$ a ball which intersects A. Then

$$A \subset B_R \subset B_{3^{k-1}(r/3)}(x_j)$$

and by the doubling condition

$$\mu(A) \le \mu(B_R) \le b^{k-1} \mu(B_{r/3}(x_j).$$

Let B_j be N such disjoint balls. The measure of there union is at least

$$N b^{1-k} \mu(B) \le \mu(\bigcup B_j) \le b\mu(B)$$

and hence there are at most b^k such balls. Take a maximal such sequence $B_{r/3}(x_i)$. Then as for the Lemma of Vitali the union of $B_r(x_i)$ covers A. □

THEOREM 5.2. *The measure μ is σ finite and regular, i.e.*

$$\mu(A) = \sup_{K \subset A} \mu(K) = \inf_{A \subset U} \mu(U) \tag{5.1}$$

for every Borel set.

PROOF. Let $x_o \in X$. We define the Borel measure

$$\mu_R(A) = \mu(B_R(x_0) \cap A).$$

It is finite and $\mu = \lim_{n\to\infty} \mu_n$, and hence it is σ finite. Let $\mathcal{A}$ be the set of Borel set for (5.1) holds. We will show that A contains all compact sets, with A it contains its complement, and it closed under countable unions. So it is a σ algebra containing all open sets. The Borel σ algebra is the smallest such σ algebra, and hence $\mathcal{A}$ consists of all Borel sets.

For compact sets K inner regularity (approximation by compacts) is trivial. Since $K = \cap U_j$ with $U_j(d(x,K) < 1/j)$ is countable intersection $\mu(K) = \lim_j \mu(U_j)$ and outer regularity follows.

A is an inner regular Borel set if and only if $X\backslash A$ is outer regular - this obvious for compact metric spaces, and requires use of σ finiteness in general.

Let $A = \bigcup A_j$ be a countable union of inner regular sets, and let $\varepsilon > 0$. Then there exists compact sets K_j and open sets U_j with

$$K_j \subset A_j \subset U_j$$

and

$$\mu(U_j) \leq \mu(K_j) + 2^{-1-j}\varepsilon.$$

Then $\bigcup_{j=1}^N K_j$ is compact and

$$\mu(\bigcup_{j=1}^N K_j) \to \mu(\bigcup_{j=1}^\infty K_j).$$

Moreover

$$\mu(A) \leq \mu(\bigcup K_j) + \varepsilon$$

and inner regularity follows. Outer regularity is similar but simpler. □

The most important such space is the Euclidean space with the Lebesgue measure.

Definition 5.3. *We call a linear operator T Calderón-Zygmund operator if*

(1) *There exists $1 < p_0 \leq \infty$ such that $T : L^{p_0}(\mu) \to L^{p_0}(\mu)$ and*

$$\|Tf\|_{L^{p_0}(\mu)} \leq A\|f\|_{L^{p_0}(\mu)}.$$

(2) *There exists a continuous kernel function $K : X \times X\backslash\{(x,x) : x \in X\}$ which satisfies*

$$Tf(x) = \int K(x,y)f(y)d\mu$$

whenever f is compactly supported and x is not contained in the support of f. Moreover

$$\int_{d(x,y)\geq 2d(y,z)} |K(x,y) - K(x,z)|d\mu(x) \leq A$$

for all $y, z \in X$.

The Hilbert transform on $\mathbb{R}$ or $\mathbb{T}$ is the most important example. It is bounded by the constant 1 as linear operator on L^2, and has the kernel $c(x-y)^{-1}$ for $\mathbb{R}$.

2. The Calderón-Zygmund theorem

We denote the doubling constant by $\mathbf{b}$ in this section.

THEOREM 5.4. *Let T be a Calderón-Zygmund operator. Then the weak type estimate holds for all $f \in L^{p_0} \cap L^1$.*

$$\mu(\{x : |Tf| > t\}) \leq (2^{p_0} + \mathbf{b}^4)\frac{A}{t}\int_X |f| d\mu$$

Let $1 < p \leq p_0$. Then T defines a unique bounded operator from $L^p(\mu)$ to $L^p(\mu)$ which satisfies

$$\|Tf\|_{L^p} \leq c(p_0) A \mathbf{b}^4 \frac{p}{p-1} \|f\|_{L^p}$$

The second statement follows from the weak type inequality by the interpolation theorem of Marcinkiewicz 2.24 or even from the simpler version in the proof of the bounds for the maximal function Theorem 2.20. The uniform constant as $p \to p_0$ involves a second application of complex interpolation Theorem 2.16.

We begin the proof by the Whitney covering lemma, first in $\mathbb{R}^n$.

A *dyadic cube* is a cube of the form

$$Q_{s,k} = 2^j([0,1)^n + k)$$

with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. A dyadic cube has 2^n children and 1 parent. Given s the cubes cover $\mathbb{R}^n$.

If $U \subset \mathbb{R}^n$ is open, and not equal to $\mathbb{R}^n$ we can cover it by dyadic disjoint cubes of size 2^s between the distance of the cube to the boundary, and 2^n times the distance to the boundary. This is called a Whitney covering.

We formalize this for spaces of homogeneous type.

LEMMA 5.5 (Whitney). *Let U be open, $U \neq X$. Then there exists a countable sequence of balls B_j so that*

1. *The balls $\frac{1}{3}B_j$ are disjoint.*
2. *The balls B_j cover U*
3. *The balls $2B_j$ are contained in U.*
4. *The balls $3B_j$ are not contained in U.*

PROOF. Given $x \in U$ we define $r(x) = d(x, X\backslash U)/3$.

The balls $B_{r(x)/3}(x)$ cover U. Let $K \subset U$ be compact. There is a finite number of the balls covering K. By the lemma of Vitali there is a disjoint subset $B_j = B_{r(x_j)/3}(x_j)$ so that the balls $B_{r(x_j)}(x_j)$ this cover K. Fix $x_0 \in X$ and let

$$K_l = \{x \in U : d(x, x_0) \leq l, d(x, X\backslash U) \leq 1/l\}$$

Then K_l is compact, monoton ($K_l \subset K_{l+1}$) and $U = \bigcup K_l$. We proceed recursively, and always keep the balls we have already chosen. □

In the sequel we denote the doubling constant by $\mathbf{b}$.

LEMMA 5.6 (Calderón-Zygmund decomposition). *Let $f \in L^1(\mu)$ and $t > \frac{1}{\mu(X)} \int |f| d\mu$. Then there exists a decomposition*

$$f = g + b$$

and

$$b = \sum_k b_k$$

so that

(1) $|g(x)| \le \mathbf{b}^2 t$, $\int |g| d\mu \le \int |f| d\mu$.
(2) b_k *is supported in balls* B_k. *They satisfy*

$$\int |b_k| d\mu \le 2t\mathbf{b}\mu(B_k), \qquad \int b_k d\mu = 0$$

(3) $\sum_k \mu(B_k) \le \frac{\mathbf{b}^2}{t} \int f d\mu$

Proof. Let

$$U = \{x : Mf(x) > t\}$$

and let B_j be the balls of the Whitney decomposition. By Theorem 2.20

$$\mu(U) \le \frac{\mathbf{b}}{t} \|f\|_{L^1}$$

We define recursively

$$Q_j = B_j \cap \{X \backslash \bigcup_{l=1}^{j-1} Q_l\}.$$

Then

$$\frac{1}{3} B_j \subset Q_j \subset B_j, \tag{5.2}$$

the Q_j are disjoint, and their union is U. We define

$$b_j = \chi_{Q_j} \left(f - \frac{1}{\mu(Q_j)} \int_{Q_j} f dx \right)$$

and

$$g(x) = \begin{cases} f(x) & \text{if } x \notin U \\ \frac{1}{Q_j} \int_{Q_j} f & \text{if } x \in Q_j \end{cases}$$

We verify the properties. It is immediate that $\int |g| d\mu \le \inf |f| d\mu$.By construction

$$f = g + b \text{ with } b = \sum b_j$$

Since

$$\int_{B_j} |f| d\mu \le \int_{3B_j} |f| d\mu \le t\mathbf{b}\mu(B)$$

and $\mu(Q_j) \ge \mathbf{b}^{-1} \mu(B_j)$ we have

$$|g| \le \mathbf{b}^2 t$$

in U. Outside U

$$|g(x)| \le Mf(x) \le t$$

almost everywhere.

Similarly

$$\begin{aligned}\int_{B_j} |b_j| d\mu =& \int_{Q_j} \left| f - \mu(Q_j)^{-1} \int_{Q_j} f d\mu \right| d\mu \\ \leq& 2 \int_{3B_j} |f| d\mu \\ \leq& 2\mathbf{b} t \mu(B_j)\end{aligned}$$

since $3B_j \cap X \backslash U \neq \{\}$, using the definition of the maximal function.

Now, by the weak type estimate of the maximal function

$$\begin{aligned}\sum \mu(B_j) \leq& \mathbf{b} \sum \mu(Q_j) \\ =& \mathbf{b}\mu(U) \\ \leq& \frac{\mathbf{b}^2}{t} \|f\|_{L^1}.\end{aligned}$$

□

We turn to the proof of weak type estimate of Theorem 5.4.

PROOF. We may choose A by multiplying T by a constant. There is nothing to show if $t \leq \mu(X)^{-1} \int_{\mathbb{R}^n} |f| d\mu$. If $t \leq \mu(X)^{-1} \int |f| d\mu$ then

$$\mu(X) \leq \int |f| d\mu / t$$

and the weak type estimate is trivial. Hence we assume $t > \mu(X)^{-1} \int |f| d\mu$. We decompose $f = g + b$ as in the Calderón-Zygmund decomposition. Then

$$\|g\|_{L^1} \leq \|f\|_{L^1}$$

by the Calderón-Zygmund decomposition. Thus

$$\begin{aligned}\mu(|T(g)(x)| > t/2) \leq& (2/t)^{-p_0} \|Tg\|_{L^{p_0}}^{p_0} \\ \leq& \left(\frac{2A}{t}\right)^{p_0} \int |g|^{p_0} d\mu \\ \leq& \frac{(2A)^{p_0}}{t} \mathbf{b}^{2p_0-2} \|g\|_{L^1} \\ \leq& \frac{(2A)^{p_0}}{t} \mathbf{b}^{2p_0-2} \|f\|_{L^1}.\end{aligned} \tag{5.3}$$

Since

$$t \sum \mu(3B_j) \leq \mathbf{b}^2 \|f\|_{L^1}$$

it suffices to bound (together with Tschebycheff's inequality)

$$\int_{X \backslash U} |Tb| d\mu \leq \sum_j \int_{X \backslash 2B_j} |Tb_j| d\mu.$$

We observe that for $x \in X \backslash 2B_j$ and z_j the center

$$Tb_j(x) = \int K(x,y) b_j d\mu = \int_{B_j} (K(x,y) - K(x,z_j)) b_j(y) d\mu(y)$$

since the $\int b_j = 0$. Thus

$$\begin{aligned}\int_{X\backslash 2B_j} |Tb_j| d\mu(x) \leq & \int\int_{d(x,z_j)\geq 2r_j} |K(x,y) - K(x,z_j)| d\mu(x)|b_j| d\mu(y) \\ \leq & A\int |b_j| d\mu \\ \leq & 2At\mathbf{b}^2\mu(Q_j).\end{aligned}$$

and hence

$$\mu(\{Tb > t/2\} \cap (X\backslash U)\} \leq \frac{2}{t}\|Tb\|_{L^1(X\backslash U)} \leq 4A\mathbf{b}^2 \sum \mu(B_j) \leq \frac{4A\mathbf{b}^4}{t}\|f\|_{L^1}$$

and hence

$$\mu(\{Tf > t\}) \leq \left(2^{p_0}A^{p_0-1}\mathbf{b}^{2(p_0-1)} + A^{-1}\mathbf{b}^2 + 4\mathbf{b}^4\right)\frac{A}{t}\|f\|_{L^1}.$$

We choose $A = \mathbf{b}^{-2}$. □

REMARK 5.7. (1) *There are obvious vector valued versions. We consider weakly continuous functions, i.e. $f : U \to E$ is weakly measurable if for every element of the dual space e∗ $e^* \circ f$ is measurable. Definition and estimate for the maximal function work without change for weakly measuable functions which assume values in Banach spaces.*

The interpolation theorems of Marcinkiewicz and Riesz-Thorin hold for operators from $L^p(\mu; E)$ to $L^q(\nu, F)$ for real resp. complex Banach spaces E and F. In that case the kernel function has values in the Banach space of continuous linear operators from E to F. Such a map to linear operators is measuable if for all $e \in E$ and $f^* \in F^*$ the map $x \to f^*(K(x,y)e$ is measurable. The integral for weakly continuous integrable functions with values in Banach spaces is defined through the application of a linear form.

(2) *If $T : L^{p_0}(\mu) \to L^{p_0}(\mu)$ then the adjoint maps $L^{p_0'}$ to $L^{p_0'}$. If $K(x,y)$ is the kernel function of T then $\overline{K(y,x)}$ is the kernel function of T^*. Thus, if*

$$\int_{d(y,x)\geq 2d(y,z)} |K(x,y) - K(z,y)| d\mu(x) \leq A$$

then T^* is a bounded operator on $L^p(\mu)$ for $1 < p \leq p_0'$. So under this condition

$$\|Tf\|_{L^p} \leq c(\mathbf{b}, p_0)\frac{p^2}{p-1}\|f\|_{L^p}.$$

LEMMA 5.8. *Suppose that there exists $\varepsilon > 0$ such that*

$$\min\{\mu(B_{d(x,y)}(x)), \mu(B_{d(\tilde{x},\tilde{y})}(\tilde{x})\}|K(x,y) - K(\tilde{x},\tilde{y})| \leq c\left(\frac{d(x,\tilde{x}) + d(y,\tilde{y})}{d(x,y) + d(\tilde{x},\tilde{y})}\right)^{\varepsilon}$$

for all $x, y, \tilde{y}, \tilde{y}$ then the kernel condition is satisfied for T and T^*. In particular, if $T : L^{p_0} \to L^{p_0}$ has an integral kernel which satisfies the condition above then it defines a unique bounded operator from L^p to L^p.

In particular this holds for the Euclidean space with the Lebesgue measure if

$$|D_x K(x,y)| + |D_y K(x,y)| \le c|x-y|^{-n-1} \tag{5.4}$$

where one of the conditions suffices for either $p \ge p_0$ or $p \le p_0$.

It is not hard to see that if T is a Calderón-Zygmund operator with integral kernal k then

$$\mu(B_{d(x,y)}(x))|K(x,y)| \le c(\mathbf{b})A \tag{5.5}$$

3. Examples

3.1. Fourier multipliers.

LEMMA 5.9 (Mihlin-Hörmander). *Let $a \in C^n(\mathbb{R}^n \backslash \{0\})$ and assume that*

$$\sup_{|\alpha|\le n+2} ||\xi|^\alpha D^\alpha a(\xi)| \le A$$

The operator

$$T : L^2 \ni f \to \mathcal{F}^{-1} a \hat{f} \in L^2$$

has a convolution kernel $K(x,y) = k(x-y)$ with

$$\sup_z |z|^n |k(z)| + |z|^{n+1}|Dk(z)| \le c(n)A$$

It is a Calderon-Zgymund operator.

PROOF. Let $\phi \in C_0^\infty(\{x : \frac{1}{2} \le |x| \le 4\})$. We claim that $k(x) = \phi(x)T$ is a differentiable function which satisfies

$$|k(x)| + |Dk(x)| \le c \tag{5.6}$$

Since the integral kernel of the Fourier multiplier $a(\lambda(x))$ is $\lambda^{-n}k(x/n)$ and since the conditions are invariant under rescaling this implies the full desired estimate. The Fourier multiplier of the convolution by ϕ is the convolution of the Fourier transform of k with a. Then $\hat{k} \in \mathcal{S}$ and

$$\int \hat{k}\xi^\alpha d\xi = 0$$

Thus, with p_{n+1} the Taylor polynomial of degree $n+1$ at ξ

$$\hat{\phi} * a(\xi) = \int \hat{\phi}(\eta)(a(\xi-\eta) - p_n(\eta))d\eta$$

and by the Taylor formula

$$|(a(\xi-\eta) - p_{n+1}(\eta))| \le c(n)|\xi|^{-n-2}|\eta|^{n+2}$$

for $|\eta| \le |\xi|/2$ and hence

$$\begin{aligned} |\phi * a(\xi)| \le & \left| \int_{|\eta|\le|\xi|/2} \hat{\phi}(\eta)(a(\xi-\eta) - p_{n+1}(\eta))d\eta \right| \\ & + c\int_{|\eta|\ge|\xi|/2} |\hat{\phi}(\eta)||\eta|^{n+2}d\xi \\ \le & c(n)(1+|\xi|^{-n-2}) \end{aligned}$$

This implies the assertion. □

The Riesz transforms defined by a Fourier multiplier are important examples:

$$a(\xi) = \xi_j |\xi|^{-1}$$

with a convolution kernel (for $n \geq 2$)

$$i\frac{2\Gamma((n+1)/2}{\pi^{\frac{n-1}{2}}} \frac{x_j}{|x|^{n+1}}$$

since

$$\mathcal{F}|x|^{-1} = \frac{\Gamma((n-1)/2}{\pi^{\frac{n-1}{2}}} |x|^{1-n}$$

Then

$$K(x,y) = i\frac{2\Gamma((n+1)/2}{\pi^{\frac{n-1}{2}}} \frac{x_j - y_j}{|x-y|^{n+1}}$$

It satisfies the conditions on the derivatives above. For $n = 1$ we obtain again the Hilbert transform. We denote the Riesz transforms by R_j.

Similarly, if $n \geq 2$, the second order Riesz transforms defined by the Fourier multiplier

$$\frac{\xi_i \xi_j}{|\xi|^2}$$

then the kernel function is

$$c_n \frac{x_i x_j - \frac{1}{n}\delta_{ij}}{|x|^{2+n}}.$$

Again this follows for $n \geq 3$ from taking derivatives of the inverse Fourier transform of $|\xi|^{-2}$, and we obtain a formula for the constant c.

The case $n = 1$ is trivial - the second order Riesz transform is the identity.

For $n = 2$ we may take a detour via the Cauchy kernel and calculate

$$\mathcal{F}^{-1} \frac{1}{\xi_1 + i\xi_2}$$

and then take the real part.

We denote the second order Riesz transforms R_{ij}. If

$$\Delta u = f$$

for $u \in \mathcal{S}$ then

$$\partial^2_{x_i x_j} u = R_{ij} f$$

and we obtain from the Calderón Zygmund estimate

$$\|\partial^2_{ij} u\|_{L^p} \leq c(n) \frac{p^2}{p-1} \|f\|_{L^p}.$$

3.2. The heat equation. The heat equation leads to a Calderón-Zygmund operator on a space of homogeneous type. Consider

$$u_t - \Delta u = f$$

If $u \in \mathcal{S}$ then

$$\hat{u}_t = \frac{i\tau}{i\tau + |\xi|^2} \hat{f}$$

and hence

$$\|u_t\|_{L^2} \leq \|f\|_{L^2}$$

and similarly

$$\|D_x^2 u\|_{L^2} \le c\|f\|_{L^2}$$

The convolution kernel is of $T : f \to u_t$ is

$$\partial_t \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} = t^{-1}(2n\pi + \frac{|x|^2}{4t}) \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

for $t > 0$ and 0 otherwise.

We define the metric

$$d((x,t),(y,s)) = \max\{|x-y|, |t-s|^{1/2}\}.$$

It is easy to verify the triangle inequality. Let μ be the Lebesgue measure. The doubling condition with $b = 3^{n+2}$ is immediate. Then

$$|g(x,t)| \le c(|x| + \sqrt{|t|})^{-n-2}$$

and

$$|\nabla_x g| \le c(|x| + \sqrt{|t|})^{-n-3}$$
$$|\partial_t g| \le c(|x| + \sqrt{|t|})^{-n-4}.$$

This implies the kernel conditions of Lemma 5.8. The L^2 boundedness follows from the Fourier transform.

3.3. Weyl quantization of symbols in $S^0_{1,0}$.

DEFINITION 5.10. *Let $m \in \mathbb{R}$, $\rho \ge \delta \in [0,1]$. We say that $a \in S^m_{\rho,\delta}(\mathbb{R}^n)$ if $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and if for all $N \in \mathbb{N}$ there exists c_N such that*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le c_{|\alpha|+|\beta|}(1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|}.$$

Let $k(w,z) = \mathcal{F}_\xi^{-1} a(w,z)$. The kernel of the standard quatization is

$$K(x,y) = k(x, x-y)$$

and the kernel of $\mathrm{Op}_t(a)$ is

$$K(x,y) = k(tx + (1-t)y, x-y)$$

Let

$$g(w,z) = k(x - (1-t)z, z)$$

and let b be the Fourier transform of g with respect to z. Then $b(x,D) = \mathrm{Op}_t\, a$. Equivalently

$$b(x,\xi) = (2\pi)^{-2n} \int e^{i\langle \eta-\xi, z\rangle} a(x-(1-t)z, \eta) d\eta dz$$

LEMMA 5.11. *Suppose that $0 \le t \le 1$ and $\rho \ge \delta \ge 0$ and $a \in S^m_{\rho,\delta}$. Then $b \in S^m_{\rho,\delta}$. Similarly, if $b \in S^m_{\rho,\delta}$ then $a \in S^m_{\rho,\delta}$,*

We only prove one direction. The reverse direction is similar.

PROOF. Let $\rho \in C^\infty(B_2(0))$, identically 1 on $B_1(0)$. Then, for all α, β and N

$$\left| \partial_x^\alpha \partial_\xi^\beta (2\pi)^{-2n} \int e^{i\langle \eta-\xi,z\rangle} \rho(\eta) a(x-(1-t)z,\eta) d\eta dz \right| \le c(1+|\xi|)^{-N}$$

by stationary phase. Now let $R > 1$ and $\rho_R(\eta) = \rho(\eta/(2R)) - \rho(\eta/R)$. Then

$$\left|\partial_x^\alpha \partial_\xi^\beta (2\pi)^{-2n} \int e^{i\langle \eta-\xi, z\rangle} \rho_R(\eta) a(x-(1-t)z, \eta) d\eta dz\right| \leq c_{N,\alpha,\beta} R^{m+\delta|\alpha|-\rho|\beta|}(1+|\xi|/R + R/|\xi|)^{-N}.$$

To see this we apply the derivatives - the x derivatives fall directly on a, the ξ derivatives after an integration by parts. This yields the factor $R^{m+\delta\alpha-\rho\beta}$. Now we change coordinates to $\tilde{x} = R^\delta x$ and $\tilde{\xi} = R^{-\delta}\xi$. In the new variables we use stationary phase. The only stationary point in the phase function is $(\xi, 0)$, and with each integration by parts we gain a power of the distance to the support.

□

Theorem 5.12. *The kernel $K(x,y)$ of $Op_t a(x,D)$ with $a \in S^0_{1,0}$ satisfies*

$$|K(x,y)| \leq C|x-y|^{-n}, \qquad |\nabla_{x,y} K(x,y)| \leq c|x-y|^{-n-1}$$

In particular $\operatorname{Op}_t a$ for $a \in S^0_{1,0}$ defines a Calderón-Zygmund operator with $p_0 = 2$. The boundedness on L^2 is a consequence of the Theorem of Calderón-Vaillancourt, and the kernel estimates (which have to be verified) imply that the assumption on the kernel of Theorem 5.4 is satisfied. It suffices to prove the bound for K $\nabla_y K$ for $t = 1$ by the previous lemma.

Proof. Let $\phi \in C^\infty(\mathbb{R}^n)$ be nonnegative, radial, supported in $B_2(0)$, indentically 1 on $B_1(0)$. Let

$$\eta_0(x) = \phi$$

and

$$\eta_j(x) = \phi(2^{j+1}x) - \phi(2^j x)$$

fpr $j \geq 1$. Then

$$\sum_j \eta_j(x) = 1$$

for $x \neq 0$. We define

$$a_j(x,\xi) = \eta_j(\xi) a(x,\xi)$$

and

$$k_j(x,z) = \left(\mathcal{F}_\xi^{-1} a_j\right)(x,z)$$

Then (this follows by several integrations by parts)

$$|\partial_x^\alpha \partial_z^\beta k_j(x,z)| \leq c(M,\alpha,\beta) 2^{jn+|\beta|}(1+2^j|z|)^{-M} \tag{5.7}$$

if $j \geq 1$ and, since a_0 is a Schwartz function with respect to ξ

$$|\partial_x^\alpha \partial_z^\beta k_0(x,z)| \leq c(M,\alpha,\beta)(1+|z|)^{-M}. \tag{5.8}$$

The case $|z| \geq 2^{-j-2}$ is the first inequality is see by scaling $\xi = 2^j\tilde{\xi}$, $x = 2^{-j}\tilde{x}$. The case $|z| \leq 2^{-j-2}$ follows by integration.

Finally

$$k(x,z) = \sum_j k_j(x,z)$$

satisfies

$$|\partial_x^\alpha \partial_z^\beta k(x,z)| \leq c|z|^{-n-|\alpha|}$$

We claim that

$$K(x,y) = k(x, x-y)$$

satisfies the Calderón Zygmund condition, as well as $K(y,x)$. Then

$$|\nabla_y K(x,y)| \leq C|x-y|^{-n-1}$$

$\square$

CHAPTER 6

Hardy and BMO

This section follows to a large extend the work of Fefferman and Stein [6] and Stein [13].

1. More general maximal functions and the Hardy space $\mathcal{H}^p$

We fix a measurable function ϕ for which there is a radial and radially decreasing majorant ϕ^*, $|\phi| \le \phi^*$. Then we have seen that

$$|\phi * f(x)| \le cMf(x).$$

We define $\phi_t(x) = t^{-n}\phi(x/t)$.

For $N \in \mathbb{N}$ we define the norm

$$\|f\|_N = \sup_{|\alpha|+|\beta|\le N} \sup_x |x^\alpha \partial^\beta f|$$

and the set of functions

$$\mathcal{F}_N = \{\phi : \|\Phi\|_N \le 1\}.$$

Definition 6.1. *We define*

$$M_\phi f(x) = \sup_t |f * \phi_t(x)|, \tag{6.1}$$

the nontangential version

$$M^*_\phi f(x) = \sup_t \sup_{|y|\le t} |f * \phi_t(x+y)| \tag{6.2}$$

and the 'grand' maximal function

$$M_N f(x) = \sup_{\phi:\|\phi\|_N\le 1} \sup_t |f * \phi_t(x)|. \tag{6.3}$$

It is important in the following theorem that we allow $p \le 1$.

Theorem 6.2. *Let f be a tempered distribution and $0 < p \le \infty$. Then the following conditions are equivalent*

1. *There exists $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \phi = 1$ so that $M_\phi f \in L^p$.*
2. *There exist seminorms N so that $M_N \phi \in L^p$*
3. $M^*_{e^{-\pi|x|^2}} f \in L^p$.

We define the *real* Hardy space $\mathcal{H}^p$ as the set of all functions for which the equivalent conditions of the Theorem hold.

If $p > 1$ then any maximal function of f majorizes a multiple of f. The second and the third are bounded by the standard Hardy-Littlewood maximal function and hence $\mathcal{H}^p = L^p$ in that case.

For $p = 1$ the same argument shows that $\mathcal{H}^1 \subset L^1$. The spaces L^p for $p < 1$ are defined in the obvious fashion. They are not Banach spaces, and they do not imbed into the space of distributions.

There are typical elements of $\mathcal{H}^p$ called atoms.

DEFINITION 6.3 (Atoms). *Let $0 < p \le 1$ A p atom is a bounded function a for which there is a ball $B = B_r(x_0)$ so that*

$$\operatorname{supp} a \subset B$$

$$|a| \le |B|^{-1/p}$$

$$\int x^\alpha a dx = 0$$

for all multiindices α with

$$|\alpha| \le n(\frac{1}{p} - 1)$$

LEMMA 6.4. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and let a be a atom with the ball $B_r(x_0)$. Then there exist ε and c so that*

$$M_\phi a \le C|B_r(x_0)|^{-1/p}(1 + \frac{|x - x_0|}{r})^{-\frac{n}{p} - \varepsilon(n,p)}$$

In particular

$$\int |M_\phi a|^p dx \le c(n,p).$$

PROOF. Exercise □

We introduce a modified nontangential maximal function for $a \ge 1$

$$M^a f(x) = \sup_t \sup_{|y| \le at} |f * \phi_t(x - y)|$$

LEMMA 6.5.

$$\|M^a f\|_{L^p} \le c a^{n/p} \|f\|_{L^p}$$

PROOF. The claim follows from

$$|\{x : M^a f > \lambda\}| \le c a^n |\{x : M_\phi^* f > \lambda\}| \tag{6.4}$$

by integration.

Let $O = \{x : M_\phi^* f > \lambda\}$. Suppose that $M^a f(x) > \lambda$. Then there exist $(\tilde{x}, \tilde{t})$ with $f * \phi_{\tilde{t}}(\tilde{x}) > \lambda$ and $|x - \tilde{x}| \le at$. Then $B_t(\tilde{x}) \subset O$ and hence

$$\frac{|O \cap B_{at}(x)|}{|B_{at}(x)|} > a^{-n}.$$

Let $A = \mathbb{R}^n \backslash O$ and

$$A^* = \{x \in A : \frac{|O \cap B_r(x)|}{|B_r(x)|} < a^{-n} \text{ for some } r\}$$

Then

$$|\mathbb{R}^n \backslash A^*| \le (3a)^n |\mathbb{R}^n \backslash A| \tag{6.5}$$

implies (6.4).

To prove (6.5) we turn to an argument in measure theory. Suppose that $A \subset \mathbb{R}^n$ is a closed set and let $0 < \gamma < 1$ $(\gamma = 1 - a^{-n})$. Let $A^* \subset A$ be the set of all points x so that

$$\frac{|A \cap B|}{|B|} \le \gamma$$

for some ball B containing x. Then

$$\mathbb{R}^n\backslash A^* = \{x : M(\chi_{\mathbb{R}^n\backslash A}) > 1-\gamma\}$$

and by the estimate for the maximal function

$$|\mathbb{R}^n\backslash A^*| \le \frac{3^n}{1-\gamma}|\mathbb{R}^n\backslash A|$$

□

For the proof of Theorem 6.2 we have to study the effect of changing the function is the definition of the maximal function. This is the easier part of the proof.

LEMMA 6.6. *Let $\phi,\psi \in \mathcal{S}$ with $\int \phi = 1$. For all $M>0$ There exists a sequence $\eta^{(k)} \in \mathcal{S}$ such that for all N*

$$\|\eta^{(k)}\|_N \le c_{N,M} 2^{-kM}$$

and

$$\psi = \sum \eta^{(k)} * \Phi_{2^{-k}}$$

PROOF. We fix $\rho \in C_0^\infty(B_2(0))$, identically 1 on B_1 and define

$$\rho_k(\xi) = \rho(2^{-k}\xi) - \rho(2^{1-k}\xi)$$

for $k \ge 1$ and $\rho_0 = \rho$. Then

$$\hat\psi = \sum_{k=0}^\infty \rho_k \hat\psi.$$

Now $1 = \int \phi dx = \hat\phi(0)$. Without loss of generality we assume $|\hat\phi(\xi)| \ge \frac12$ for $|\xi| \le 2$. Then

$$\hat\psi(\xi) = \sum_{k=0}^\infty \frac{\rho_k(\xi)}{\hat\phi(2^{-k}\xi)}\hat\Psi(\xi)\hat\phi(2^{-k}\xi) = \hat\eta^{(k)}\hat\phi(2^{-k}\xi)$$

Now $\hat\Psi$ is a Schwartz function which leads to the claimed decay. □

The proof gives actually a stronger statement: Given M there exists N so that the claim holds for $\|\psi\|_N < \infty$.

We turn to the proof of the theorem.

PROOF. Let $\Phi \in \mathcal{S}$ with $\int \Phi = 1$. We claim that there is are constants c and N so that

$$\|\mathcal{M}_N f\|_{L^p} \le c\|M^*_\Phi f\|_{L^p} \tag{6.6}$$

and

$$\|M^*_\Phi f\|_{L^p} \le c\|M_\Phi f\|_{L^p} \tag{6.7}$$

These two estimates imply all assertions of the theorem.

To proof the first inequality we choose $\psi \in \mathcal{S}$. The expansion gives

$$M_\psi f(x) = \sup_t |f * \psi_t(X)| \le \sup_{t>0} \sum_{k=0}^\infty |f * \Phi_{2^{-k}t} * \eta_t^{(k)}(x)|$$
$$\le \sup_t t^{-n} \sum_k \int |f * \Phi_{2^{-k}t}(x-y)||\eta^{(k)}(y/t)|dy$$
$$\le \sup_t \sum_k \sup_y |f * \Phi_{2^{-k}t}(x-y)|(1+\frac{|y|}{2^{-k}t})^{-N} \int t^{-n}(1+\frac{|y|}{2^{-k}t})^N |\eta^{(k)}(y/t)|dy$$

with $N > n/p$ and

$$t^{-n}\int (1+\frac{|y|}{2^{-k}t})^N |\eta^{(k)}(y/t)|dy \le c2^{-k}$$

if (which is ensured by Lemma 6.6)

$$\|\eta^{(k)}\|_{N+n} \le c2^{-k(N+1)}.$$

We claim that

$$\| \sup_t \sup_y |f * \Phi_t(x-y)(1+\frac{|y|}{t})^{-N} \|_{L^p} \le c\|M_\phi^* f(x)\|_{L^p}. \tag{6.8}$$

Then

$$\sup_t \sup_y |f * \Phi_t(x-y)(1+\frac{|y|}{t})^{-N} \le \sup_{j=0,1\dots} 2^{-jN} M^{2^j} f(x)$$

and the assertion follows Lemma 6.5.

This implies (6.6).

To complete the proof we will prove

$$\|M_\phi^* f\|_{L^p} \le c\|f\|_{L^p}. \tag{6.9}$$

Let

$$F_\lambda = \{x : M_N f(x) \le \lambda M_\phi^* f(x)\}.$$

Since

$$\int_{\mathbb{R}^n\setminus F} |M_\Phi^* f|^p dx \le \lambda^{-p} \int_{\mathbb{R}^n\setminus F} |M_N f|^p dx \le c^p \lambda^{-p} \int |M_\Phi^* f|^p dx$$

we obtain

$$\int_{\mathbb{R}^n} |M_\Phi^* f|^p dx \le 2 \int_F |M_\Phi^* f|^p dx$$

provided we choose $\lambda^p \ge 2c^p$.

We claim that on F and any $q > 0$

$$M_\Phi^* f(x) \le c[M|M_\Phi f|^q(x)]^{1/q}. \tag{6.10}$$

This implies the desired estimate via

$$\int_{\mathbb{R}^n} |M_\Phi^* f|^p dx \le 2\int_F |M_\Phi^* f|^p dx \le c_1 \int [M|M_\Phi f|^q]^{p/q} dx \le c_2 \int |M_\Phi f|^p dx$$

by the estimate for the Hardy-Littlewood maximal function. It remains to prove (6.10). Let

$$f(x,t) = f * \Phi_t(x), \qquad f^*(x) = M_\Phi^* f(x)$$

By definition, for any x there exists (y,t) with $|x-y|\le t$ so that

$$|f(y,t)|\ge f^*(x).$$

By the fundamental theorem of calculus

$$|f(x',t)-f(y,t)|\le rt\sup_{|z-y|<rt}|D_xf(z,t)|$$

for $|x'-y|\le rt$. However

$$\partial_{x_i}f(z,t)=\frac{1}{t}f*(\partial_i\Phi)_t(z)$$

hence

$$|f(x',t)-f(y,t)|\le cfM_Nf(x)\le cr\lambda M^*_\Phi f(x)=c\lambda rf^*(x)$$

if $x\in F$. We take r so small that $c\lambda r\le\frac{1}{4}$ to achieve

$$|f(x',t)|\ge\frac{1}{4}f^*(x)\tag{6.11}$$

for $|x'-y|\le rt$. Thus

$$\begin{aligned}|M^*_\Phi f(x)|^q\le&\left(\frac{1+r}{r}\right)^n\frac{4^q}{|B_{(1+r)t}(x)|}\int_{B_{x,(1+r)t}}|f(x',t)|^qdx'\\ \le&cM[(M_\Phi f)^q](x).\end{aligned}$$

The second inequality follows from

$$|f(x',t)|\le M_\phi(x')$$

and the first from the lower bound (6.11).

There is a last tricky part: We severely used that $\|M^*_\phi f(x)\|_{L^p}<\infty$. To deal with that we repeat the arguments with

$$M^{\varepsilon,L}_\Phi f(x))=\sup_{|x-y|<t<\varepsilon^{-1}}|f*\Phi_t(y)|\frac{t^L}{(\epsilon+t+\epsilon|y|)^L}$$

instead of M^*. If f is a tempered distribution we choose L large and ε small so that $\|M^{\varepsilon,L}_\Phi f\|_{L^p}<\infty$

Then we introduce the factor

$$\frac{t^L(\varepsilon+2^{-k}t+\varepsilon|x-y|)^L}{\varepsilon+t+\varepsilon|x|)^{-L}(2^{-k}t)^{-L}(1+\frac{2^k|y|}{t})^N}\le c2^{kL}(1+\frac{|y|}{t})^L(1+\frac{2^k|y|}{t})^N$$

as suitable points. We complete the proof as above.

□

2. The atomic decomposition

The key part of the proof is a refined Calderón-Zygmund decomposition.

We recall that we can write any nonempty set $U\subset\mathbb{R}^n$, $U\ne\mathbb{R}^n$ as the union of dyadic cubes

$$Q_{kl}=2^l([0,1)^n+k)$$

such that the length of the edge is at least the distance to the complement, and at most n times the distance. We fix two numbers $1<a<b<1+1/(4n)$ and denote $\tilde Q=aQ$, $Q^*=bQ$ where aQ resp bQ denotes the cube Q scaled by a with center the center of the cube.

PROPOSITION 6.7. *Let $f \in L^1_{loc}$ with $M_{e^{-2\pi|x|^2}} f \in L^1$ and $\lambda > 0$. Then there is a decomposition*

$$f = g + b, b = \sum b_k$$

and a collection of dyadic cubes Q_k so that

(1) $|g| \le c(n)\lambda$
(2) $\operatorname{supp} b_k \subset Q_k^*$ *and* $\int b_k dx = 0$
(3) *The Q_k are disjoint and*

$$\bigcup Q_k = \{x : M_N f > \lambda\}$$

PROOF. We fix $\zeta \in C_0^\infty(((0,1)^n)^*)$, identically one on $[0,1]^n$. For $k \in \mathbb{Z}^n$ and $l \in \mathbb{Z}$ we define

$$\zeta_{kl} = \zeta(2^{-l}x - k)$$

which is supported in Q_{kl}^* and identically 1 in $\tilde{Q}_{kl}$.

Let $O = \{x : M_N f(x) > \lambda\}$, let Q_{k_j,l_j} be a Whitney decomposition (with disjoint cubes, and edge lengths comparable to the distance to the complement), and

$$\eta_j = \frac{\zeta_{k_j,l_j}}{\sum_i \zeta_{k_i,l_i}}$$

a partition of unity. Then, if l_j is the edge length,

$$|\partial^\alpha \eta_j| \le c 2^{-l_j|\alpha|}.$$

We define

$$b_j = (f - c_j)\eta_j, \qquad c_j = \frac{\int f \eta_j dx}{\int \eta_j}$$

Then, by the definition of Q_j there exist r and $x \in \mathbb{R}^n \backslash O$ such that

$$Q_j \subset B_r(x_0)$$

and $r \le c(n)|Q_j|^{1/n}$. But then

$$\|\eta_j(\frac{x - x_0}{r})\|_N \le c(n)$$

and

$$\left| \int \eta_j f dx \right| \le c(n) r^n M_N f(x_0) \le c(n) r^n \lambda.$$

Thus

$$|c_j| \le c(n)\lambda.$$

Now

$$|g| \le c M_N f(x) \le c\lambda$$

for $X \notin O$. Together this gives the bound on g. □

THEOREM 6.8. *Suppose that $0 < p \le 1$ and $f \in \mathcal{H}^p$. Then there exists a sequence of p atoms a_j and a summable sequence λ_j so that*

$$f = \sum \lambda_j a_j$$

and

$$\sum |\lambda_j|^p \le c(n,p) \|M_{e^{-\pi|x|^2}} f\|_{L^p}^p.$$

REMARK 6.9. *Since*

$$|M_\phi \sum_j \lambda_j a_j|^p \le \sum_j |\lambda_j|^p |M_\phi a_j|$$

$$= \sum_j |\lambda_j||M_\phi a_j|^p$$

any such sum is bounded in $\mathcal{H}^p$. *The sum*

$$\sum_{\lambda_j} a_j$$

converges in the space of tempered distributions.

PROOF. We only consider $p = 1$. We have seen that $\mathcal{H}^1 \subset L^1$. Let $f \in \mathcal{H}^1$. It is integrable. For each integer l we apply the Calderón-Zygmund decomposition at level 2^l and we write $f = g^l + b^l$, $b^l = \sum_j b^l_j$.

We claim that

$$g^l \to f$$

in $\mathcal{H}^1$ for $l \to \infty$, or, equivalently, $\|b^l\|_{\mathcal{H}^1} \to 0$ as $l \to \infty$. This follows from

$$\begin{aligned}
\|b^l\|_{\mathcal{H}^1} &\sim \int M_{e^{-\pi|x|^2}} b^j dx \\
&\le \sum_j \int M_{e^{-\pi|x|^2}} b^j_k dx \\
&\le \int_{\bigcup Q^l_j} (M_N f) dx \\
&= \int_{M_N f > 2^l} M_N f dx \to 0
\end{aligned}$$

Since $|g^l| \le c2^l$ we have $g^l \to 0$ as $l \to -\infty$ in the sense of distributions. Hence

$$f = \sum_l g^{l+1} - g^l = \sum_l b^l - b^{l+1}$$

in the sense of distributions. The difference $g^{l+1} - g^l$ is supported in

$$O^l = \{x : M_n f > 2^k\}$$

and

$$g^{l+1} - g^l = b^l - b^{l+1} = \sum_j (f - c^l_j)\eta^l_j - \sum_j (f - c^{l+1}_j \eta^{l+1}_j) = \sum_j A^l_j$$

with

$$A^l_j = (f - c^l_j)\eta^l_j - \sum_m (f - c^{l+1}_m)\eta^{l+1}_m \eta^l_j + \sum_m c_{j,m}\eta^{l+1}_m$$

with

$$c_{j,m} = \frac{\int (f - c^{l+1}_m \eta^l_j)\eta^{l+1}_m dx}{\int \eta^{l+1}_m dx},$$

since η_j^l is a partition of unity and hence

$$\sum_j c_{j,m} = 0.$$

Then

$$\int A_j^l dx = 0$$

$$\operatorname{supp} A_j^l \subset \tilde{Q}_j^{l,*}$$

$$|A_j^l| \le c2^l$$

by construction. We set

$$a_j^l = c^{-1}2^{-l}|Q_j^l|^{-1}A_j^l$$

and

$$\lambda_j^l = c2^l|Q_j^l|.$$

Then the a_j^l are atoms, and

$$\sum \lambda_j^l = c\sum 2^l|Q_j^l| = c\sum_l 2^l|\{M_N f > 2^l\}| \le c\int M_N f dx.$$

□

It is not hard to see that $\mathcal{H}^1$ is a Banach space. We can use the atomic decomposition to define a norm:

$$\|f\|_{\mathcal{H}^1} = \inf\{\sum |\lambda_k| : \text{ there exists atoms with } f = \sum \lambda_k a_k\}. \tag{6.12}$$

It is a consequence that the span of atoms is dense in $\mathcal{H}^1$.

Corollary 6.10. *Let T be a Calderón-Zygmund operator. Then T defines a unique continuous operator from the Hardy space $\mathcal{H}^1$ to L^1. If T is a convolution operator satisfying the assumptions of the Mihlin-Hörmander theorem then T defines a unique continuous operator on $\mathcal{H}^1$.*

Proof. We only prove the first part. The second part is an exercise. Let a be an atom. We want to prove that

$$\|Ta\|_{L^1} \le c(n).$$

By translation invariance we may assume that the corresponding ball has center 0, and be rescalling we may assume that the radius is 1. Then

$$\|Ta\|_{L^{p_0}} \le c\|a|_{L^{p_0}} \le c(n)$$

where p_0 is the exponent of the Calderón-Zygmund operator. We use this bound on $B_2(0)$. Outside we argue as for the proof of the boundedness of Calderón-Zygmund operators.

Now let $f \in \mathcal{H}^1$. By the atomic decomposition

$$f = \sum \lambda_j a_j$$

We define

$$Tf = \sum \lambda_j T a_j$$

where the right hand side converges in L^1. There is no other choice for the definition, but wellposedness has to be proven. Suppose

$$f = \sum \lambda_j a_j = \sum \mu_j b_j$$

with atoms $(a_j), (b_j)$ and summable sequences λ_j and μ_j. We have to show that for $\varepsilon > 0$ there exists N_0 so that for all $t > 0$ and $N > N_0$

$$|\{x : |\sum_{j=1}^{N} \lambda_j T a_j - \sum_{j=1}^{N} \mu_j T b_j| > t\}| < \varepsilon/t. \tag{6.13}$$

Then

$$\sum_{j=1}^{\infty} \lambda_j T a_j = \sum_{j=1}^{\infty} \mu_j T b_j$$

follows. Inequality (6.13) follows from two properties:

(1) The weak type inequality for Calderón Zygmund operators

$$|\{x : |Tg(x)| > t\}| \le \frac{c}{t} \|g\|_{L^1}$$

(2) The convergence of the partial sums in L^1.

By the convergence there exists for $\tilde\varepsilon > 0$ and N_0 so that

$$\|\sum_{j=1}^{\infty} \lambda_j a_j dx - \sum_{j=1}^{N} \lambda_j a_j \|_{L^1} + \|\sum_{j=1}^{\infty} \mu_j b_j dx - \sum_{j=1}^{N} \mu_j b_j \|_{L^1} < \tilde\varepsilon$$

and then

$$|\{x : |\sum_{j=1}^{N} \lambda_j T a_j - \sum_{j=1}^{N} \mu_j T b_j| > t\}| < c\varepsilon'/t.$$

□

3. Duality and BMO

Definition 6.11. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. The sharp maximal function is defined by*

$$f^\sharp(x) = \sup_{B_r(y) \ni x} |B_r(y)|^{-n} \int_{B_r(y)} \left| f(w) - |B_r(y)|^{-1} \int_{B_r(y)} f(z) dz \right| dw \in [0, \infty]$$

Properties

(1) $(f+g)^\sharp(x) \le f^\sharp(x) + g^\sharp(x)$
(2) $f^\sharp(x) \le 2Mf(x)$. Hence $\|f^\sharp\|_{L^p} \le c_n \frac{p}{p-1} \|f\|_{L^p}$

Definition 6.12. *We define BMO as the space of all function for which the (semi) norm*

$$\|f\|_{BMO} = \|f^\sharp\|_{sup}$$

is finite.

Certainly $L^\infty \subset BMO$. Moreover $\ln(|x|) \in BMO$.

THEOREM 6.13. *Let $L : \mathcal{H}^1 \to \mathbb{R}$ be a continous linear map. Then there exists $f \in BMO$ such that for every atom*

$$L(a) = \int afdx, \tag{6.14}$$

$$\|f\|_{BMO} = \|L\|_{(\mathcal{H}^1)^*}.$$

Vice versa: let $f \in BMO$. Then (6.14) *defines a continuous linear functional on $\mathcal{H}^1$.*

PROOF. We prove the second part first. If a is an atom with ball B then

$$\int fadx = \int_B (f - f_B)adx \le \|f - f_B\|_{L^1(B9)}\|a\|_{L^\infty} \le f^\sharp(x_0).$$

thus for $f \in L^\infty$ and $g \in \mathcal{H}^1$ or $f \in BMO$ and $g \in \mathcal{H}^1_a$

$$\int fgdx \le \|f\|_{BMO}\|g\|_H^1$$

This implies the second statement.

Now let L be a linear function on $\mathcal{H}^1$ of norm at most 1. let B be a ball. Then

$$L^2(B) \ni f \to L(f - f_B)$$

defines a linear functional on L^2 which is represented by a function g^B so that

$$L(f - f_B) = \int g^B f dx$$

and in particular $\int_B g^b = 0$. We search for a function g so that

$$g - g_B = g^B$$

for all balls B. Let $B \subset B'$ be two balls and g^B resp. $g^{B'}$ the functions constructed above. For $f \in L^2(B)$ with $\int fdx = 0$ we have

$$L(f) = \int_B fg^B dx = \int_{B'} fg^{B'} dx$$

thus for such f

$$\int_B (g^B - g^{B'})fdx = 0.$$

Thus $g^B - g^{B'}$ is constant on B. We define

$$g = g^{B_1(0)}$$

if $|x| < 1$. Choose

$$c_R = g^{B_R(0)} - g$$

for $x \in B_1(0)$ and define

$$g(x) = g_{B_R(0)}(x) - c_R$$

for $|x| < R$ and $R \ge 1$. This gives a consistent choice, and by the considerations of the first part $g \in BMO$. □

The following theorem has been proven by different methods by John and Nirenberg [8].

THEOREM 6.14 (John-Nirenberg inequality). *There exists $\delta > 0$ such that for all balls B and all functions $f \in BMO$*

$$\int_B e^{\delta \frac{|f(x)-f_B|}{\|f\|_{BMO}}} dx \le c|B|$$

PROOF. This is a consequence of the previous proof and the bound for the maximal function. We assume that $\|g\|_{BMO} \le 1$. If $p > 1$ and $g \in L^{p'}(B)$ with $\int_B g = 0$ then

$$\|g\|_{\mathcal{H}^1} \le c\frac{p'}{1-p'}|B|^{1-\frac{1}{p'}}\|g\|_{L^{p'}} = p|B|^{\frac{1}{p}}\|g\|_{L^{p'}}$$

Then

$$|B|^{-1}\|f - f_B\|_{L^p}^p \le cp^p\|g\|_{BMO}^p$$

and by Tschebychev

$$|\{x \in B : |f - f_B| > \lambda\}| \le \frac{(cp)^p}{\lambda^p}|B|.$$

If $\lambda > \frac{1}{2c}$ we choose $p = \lambda/(2c)$ and get

$$|\{x \in B : |f - f_B| > \lambda\}| \le \left(\frac{1}{2}\right)^{\frac{\lambda}{2c}} |B| \le e^{-\delta\lambda}|B|.$$

The assertion follows by integration with respect to λ. □

THEOREM 6.15. *Let $1 < p_0 < \infty$, $T : L^{p_0}(\mathbb{R}^n) \to L^{p_0}(\mathbb{R}^n)$ linear and continuous with an integral kernel $K(x,y)$ which satisfies*

$$\int_{\mathbb{R}^n \setminus B_{2|y-\tilde{y}|}(y)} |K(x,y) - K(\tilde{x},y)|dy \le A$$

Then

$$\|Tf\|_{BMO} \le c\|f\|_{L^\infty}$$

PROOF. We fix a ball $B = B_r(x_0)$ and $\omega = |B|$. We decompose $f = f_1 + f_2 = \chi_{B_{2r}(x_0)}f + \chi_{\mathbb{R}^n \setminus B_{2r}(x_0)}f$ and use Jensen's inequality to control the first term.

$$\begin{aligned}\omega^{-1}\int\left|Tf_1 - \omega^{-1}\int Tf_1\right| \le &\left(\omega^{-1}\int|Tf_1|^{p_0}\right)^{\frac{1}{p_0}} \\ \le &A\omega^{-1/p_0}\|f_1\|_{L^{p_0}} \\ \le &\|f\|_{L^\infty}\end{aligned}$$

Since for $x, \tilde{x} \in B_1$

$$\begin{aligned}|Tf_2(x) - Tf_2(\tilde{x})| = &\left|\int_{\mathbb{R}^n \setminus B_R(x_0)}(K(x,y) - K(\tilde{x},y))f(y)dy\right| \\ \le &\int_{\mathbb{R}^n \setminus B_{2R}(x_0)}|K(x,y) - K(\tilde{x},y)|dy\|f\|_{L^\infty} \\ \le &A\|f\|_{L^\infty}.\end{aligned}$$

□

THEOREM 6.16. *The inequality*

$$\int fg dx \le c \int f^\sharp M_N g dx$$

holds whenever $g \in \mathcal{H}^1$ *and* f *is bounded.*

PROOF. We apply the atomic decomposition, together with its proof. We have

$$g = \sum \lambda_{j,l} a_j^l$$

Then

$$\begin{aligned}\int fg dx =& \sum \lambda_{j,l} \int_{\tilde{Q}_j^l} f a_k^j dx \\ =& \sum_{j,l} \int_{\tilde{Q}_j^l} (f - f_{B_j^l}) a_k^j dx \\ \le& \sum_{j,l} \frac{\lambda_{j,l}}{|Q_j^l|} \int_{Q_j^l} f^\sharp(x) dx \\ =& c \sum_l 2^k \int_{M_N G > 2^l} f^\sharp(x) dx \\ \le& c \int f^\sharp M_n g dx\end{aligned}$$

□

COROLLARY 6.17. *Suppose that* $1 < p < \infty$. *Then*

$$\|f\|_{L^p} \le cp \|f^\sharp\|_{L^p}$$

PROOF. Suppose that $f \in L^p \cap L^\infty$. Then

$$\begin{aligned}\|f\|_{L^p} =& \sup_{\|g\|_{L^{p'}} \le 1} \int fg dx \\ =& \sup_{g \in \mathcal{H}^1, \|g\|_{L^{p'}} \le 1} \int fg dx \\ \le& c_n \sup_{g \in \mathcal{H}^1, \|g\|_{L^{p'}} \le 1} \int f^\sharp M g dx \\ \le& c_n \sup_{g \in \mathcal{H}^1, \|g\|_{L^{p'}} \le 1} \|f^\sharp\|_{L^p} \|Mg\|_{L^{p'}} \\ \le& c_n p \|f^\sharp\|_{L^p}.\end{aligned}$$

□

4. Relation to harmonic functions

We consider harmonic functions u on the upper halfplane $\{x_{n+1} > 0\}$ in $\mathbb{R}^{n+1}$. We denote the coornate $x_{n+1} = t$.

If u is such a function we define

$$u^*(x) = \sup_{t>0} |u(x,t)|$$

with $x \in \mathbb{R}^n$.

THEOREM 6.18. *The following is equivalent.*

(1) $u^* \in L^p$
(2) *There exists* $f \in \mathcal{H}^p$ *so that*
$$u(t,x) = P_t * f$$
where
$$P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}$$
is the Poisson kernel.

PROOF. We restrict the proof to $p > \frac{n}{n+1}$. We observe that
$$P_t(x) = t^{-n} \frac{1}{(1 + (|x|/t)^2)^{\frac{n+1}{2}}} =: t^{-n} \phi(x/t)$$
satisfies
$$|x^\alpha \partial_x^\beta \phi| \le c_{\alpha\beta}$$
and hence, with $u(t,x) = P_t * f$
$$\|u^*\|_{L^p} \le c \|M_{e^{-\pi|x|^2}} f\|_{L^p}$$
A closer check shows that the condition $p > \frac{n}{n+1}$ is not needed.

Now suppose that $u^* \in L^p$. We claim that with $f_\varepsilon = u(\varepsilon, x)$
$$P_t * f_\varepsilon = u(\varepsilon + t, x)$$
Both functions are bounded and harmonic and they coincide on $\mathbb{R}^n$. By Liouville's theorem they are equal. Thus the family f_ε is uniformly bounded in $\mathcal{H}^p$. Hence there is a sequence f_{ε_j} converging to some f in the sense of distributions. But then
$$u(t,x) = P_t * f$$
and $f \in \mathcal{H}^p$. □

PROPOSITION 6.19 (Analog of Cauchy-Riemann). *Suppose that*
$$F = (u_0, u_1, \dots u_n)$$
are harmonic functions which satisfy the Cauchy-Riemann type equations
$$\partial_t u_0^\varepsilon + \sum_{j=1}^n \frac{\partial u_j^\varepsilon}{\partial_{x_j}} = 0, \qquad \frac{\partial u_j}{\partial x_i} = \frac{\partial u_i}{\partial x_j}.$$
Then
$$\| \sup_{t>0} |u_0(x,t)| \|_{L^1} \le c \sup_t \|F(.,t)\|_{L^1}.$$

PROOF. The key fact is that
$$|F|^q$$
is subharmonic for $q > \frac{n-1}{n}$ as in Chapter 3 for $n = 2$. The claim follows as in Theorem 3.16 based on Proposition (3.14). □

THEOREM 6.20. *Let* $f \in L^1$. *Then* $f \in \mathcal{H}^1$ *iff* $R_j f \in L^1$ *for all Riesz transforms.*

REMARK 6.21. *The Riesz transforms define a distribution, which we assume to be in* L^1.

PROOF. We define

$$F_\varepsilon = (u_0^\varepsilon, u_1^\varepsilon, \dots u_n^\varepsilon)$$

where

$$u_0^\varepsilon(t,x) = f^\varepsilon * P_t$$
$$u_j^\varepsilon(t,x) = f^\varepsilon * Q_t^j$$

with

$$Q_1^j = \frac{c_n x_j}{(1+|x|^2)^{\frac{n+1}{2}}}.$$

Then

$$\partial_t u_0^\varepsilon + \sum_{j=1}^n \frac{\partial u_j^\varepsilon}{\partial_{x_j}} = 0, \qquad \frac{\partial u_j}{\partial x_i} = \frac{\partial u_i}{\partial x_j}$$

and

$$|F_\varepsilon(x,t)| \le |F_\varepsilon(.,0)| * P_t(x)$$

Since

$$F_\varepsilon(x,0) = (f * \phi_\varepsilon, R_j f * \phi_\varepsilon)$$

we get

$$\| |F_\varepsilon| \|_{L^1} \le c$$

By Fatou

$$\sup_{t>0} \int_{\mathbb{R}^n} |F(x,t)| dx \le c$$

□

5. Div-curl type results

Here we follow Coifman, Lions, Meyer and Semmes [3, 4].

COROLLARY 6.22 (div-curl lemma 1). *Let* $n \ge 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$, $f, g \in L^p(\mathbb{R}^n; \mathbb{R}^n)$

$$div\, f \in L^p, \qquad curl\, g \in L^q.$$

Then

$$\langle f, g \rangle \in \mathcal{H}^1$$

PROOF. We fix $\phi \in C_0^\infty(\mathbb{R}^n)$ supported in $B_2(0)$, idntically 1 on $B_{1/2}(0)$ with $\int \phi dx = 1$. Then $g = \nabla G$ for some function g and

$$\langle f, g \rangle \langle f, \nabla G \rangle = \nabla \cdot (Gf)$$

and

$$\begin{aligned} \nabla \cdot (Gf) \phi_t(x) = & - t^{-1} \int Gf(x-y)(\nabla \phi)_t(y) dy \\ \le & t^{-1-n} \|G\|_{L^r(B_{2t}(x))} \|f\|_{L^{r'}(B_{2t}(x))} \\ \le & t^{-1-n} \|g\|_{L^{\tilde r}(B_{2t}(x))} \|f\|_{L^r(B_{2t}(x))} \\ \le & 2^n (M|g|^{\tilde r}(x))^{1/\tilde r} (M|f|^{r'}(x))^{1/r'} \end{aligned}$$

hence

$$M_\phi \langle f, g \rangle (x) \le 2^n \|M|g|^{\tilde r}\|_{L^{p/\tilde r}}^{1/\tilde r} \|M|f|^{r'}\|_{L^{q/r'}}^{1/r'} \le c \|g\|_{L^p} \|f\|_{L^q}$$

provided

$$1 < \tilde{r} < p, 1 < r' < q, \frac{1}{\tilde{r}} + \frac{1}{r'} = 1 + \frac{1}{n}$$

□

COROLLARY 6.23. *Let f and g as above, $n = 2$ and*

$$\Delta u = \langle f, g \rangle.$$

Then u is continuous.

COROLLARY 6.24. *Let $u : \mathbb{R} \to \mathbb{R}^n$ satisfy $\partial_j u^i \in L^n$. Then*

$$\det(Du) \in \mathcal{H}^1$$

PROOF. We expand the determinant with respect to the first row. Then

$$\det Du = \langle \nabla u, F \rangle$$

where F is given by subdeterminants. For smooth function

$$\partial_i F^i = 0$$

(see Evans, Partial Differential Equations, Theorem 2 in Section 8.1) and hence this is true for the functions at hand. The statement follows now from the previous assertion. □

REMARK 6.25. *If $f \in \mathcal{H}^1$ is nonnegative on $B_1(0)$ then*

$$\int_{B_{1/2}(0)} f |\ln f| dx \leq c$$

Hence the determinant has some higher integrability if it is nonnegative.

Let (u, p) be a solution to the Navier-Stokes equations. Then

$$\Delta p = -\sum_{i,j=1}^{n} (\partial_j u^i)(\partial_i u^j)$$

and $\sum_{i=1}^n \partial_i u^i = 0$. We fix i. Then

$$\operatorname{curl} \nabla u^i = 0 \qquad \text{and } \nabla \cdot \partial_i u = 0$$

and hence

$$\Delta p \in \mathcal{H}^1$$

Then, if $n = 2$

$$\|p\|_{sup} \leq c \|\nabla u\|_{L^2}^2$$

6. Application to elliptic PDEs

Let $U \subset \mathbb{R}^n$ be open. We denote by $H^1(U)$ the set of all functions in $L^2(U)$ with distributional derivatives in L^2, with norm

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \||Du|\|_{L^2}^2.$$

It is a closed subspace of the Hilbert space $(L^2)^{n+1}$ and hence a Hilbert space with inner product

$$\langle u, v \rangle = \int u\bar{v} + \sum_j \partial_j u \partial_j v dx$$

We denote by $H^1_0(U)$ the closure of $C_0^\infty(U)$. It encodes the boundary value 0. We assume $n \ge 3$ and we will rely on two properties for balls $B = B_r(x_0)$, the Sobolev inequality

$$\|u\|_{L^{\frac{2n}{n-2}}(B)} \le c\||Du|\|_{L^2(B)} \qquad \text{for } u \in H^1_0(B_1) \tag{6.15}$$

and the composition with Lipschitz functions g with $g(0) = 0$,

$$\|g \circ u\|_{H^1} \le \|g\|_{Lipschitz}\|u\|_{H^1}. \tag{6.16}$$

Let $(a^{ij})_{1\le i,j\le n}$ be real functions in $L^\infty(U)$ such that there exists $\kappa > 0$ with

$$\sum_{i,j} a^{ij}\xi_i\xi_j \ge \kappa|\xi|^2$$

for almost every x and every $\xi \in \mathbb{R}^n$.

Let $f, F^j \in L^2(U)$. We call u a weak solution to

$$\sum_{i,j} \partial_i a^{ij}\partial_j u = \sum_i \partial_i f^i + g$$

if this equation holds in the sense of distributions. In that case it holds in $(H^1_0)^*$ - i.e. we may test it with functions in H^1_0 instead of C_0^∞. If U is bounded then the Lemma of Lax-Milgram and the Poincaré inequality imply existence of a unique weak solution $u \in H^1_0(U)$.

Theorem 6.26. *There exists C such that the following is true. Let $u \in H^1(B_1(0))$ be a nonnegative weak solution to*

$$\sum_{i,j=1}^n \partial_i a^{ij}\partial_j u = 0$$

in a ball B. Then the Harnack inequality

$$\sup_{x\in\frac{1}{2}B} u \le C \inf_{x\in\frac{1}{2}B} u.$$

holds.

This has been proven by different methods by De Giorgi [5] and Nash [12, 11]. We follow the proof of Moser [9].

Proof. We begin with the preliminary L^∞ estimate with a technique known as Moser iteration.

Lemma 6.27. *There exists c depending only on n, $\|a^{ij}\|_\infty/\kappa$ such that the following is true. Suppose that u is a solution on $B_1(0)$. Then*

$$\|u\|_{L^\infty(B_{1/2}(0))} \le c\|u\|_{L^2(B_1(0))}$$

Proof. Given $\frac{1}{2} \le r < R \le 1$ let

$$\eta(x) = \begin{cases} 1 & \text{if } |x| \le r \\ 0 & \text{if } |x| \ge R \\ \frac{R-|x|}{R-r} & \text{if } r \le |x| \le R \end{cases}$$

Then, neglecting that $f(u) = |u|^{2(p-1)}u$ is not Lipschitz (this can be fixed by truncating at height H, and letting $H \to \infty$,

$$\begin{aligned}
0 =& \int_{B_1(0)} a^{ij}\partial_i u \partial_j(\eta^2|u|^{2(p-1)}u)dx \\
=& \frac{2p-1}{p^2}\int_{B_1(0)} a^{ij}\partial_i(\eta|u|^{p-1}u)\partial_j(\eta|u|^{p-1}u)dx \\
& - (2\frac{2p-1}{p^2} - \frac{2}{p})\int a^{ij}(\partial_i\eta)\partial_j(\eta|u|^{p-1}u)|u|^{p-1}udx \\
& - (\frac{2}{p} - \frac{2p-1}{p^2})\int (a^{ij}\partial_i\eta\partial_j\eta)|u|^{2p}dx \\
=& \frac{2p-1}{p^2}\int a^{ij}\partial_i v\partial_j v dx \\
& - \frac{2(p-1)}{p^2}\int a^{ij}\partial_i v(\partial_j\eta)|u|^{p-1}udx \\
& - \frac{1}{p^2}\int a^{ij}(\partial_i\eta)(\partial_j\eta)|u|^{2p}dx
\end{aligned}$$

and hence, with $v = \eta|u|^{p-1}u$,

$$\|u\|^{2p}_{L^{\frac{2pn}{n-2}}(B_r(0))} \le c\|v\|^2_{H^1} \le \frac{c}{(R-r)^2}\|u\|^{2p}_{L^{2p}(B_R(0))} \tag{6.17}$$

For $k \in \mathbb{N}$ we write $p_k = \left(\frac{n}{n-2}\right)^k$, and for $1 \le j \le k$ we set $r_j = 1 - \frac{1}{2j}$. Then

$$\begin{aligned}
\|u\|_{L^{2p_k}(B_{r_k})} \le& \left(c(k-1)^2\right)^{1/p_k}\|u\|_{L^{2p_{k-1}}(B_{r_{k-1}})} \\
\le& \prod_{j=1}^{k}(cj^2)^{(\frac{n-2}{n})^j}\|u\|_{L^2(B_1)}.
\end{aligned}$$

Since

$$\prod_{j=1}^{k}(cj^2)^{(\frac{n-2}{n})^j} = \exp\left(\sum_{i=1}^{j}(\ln c + 2\ln i)(\frac{1}{\frac{n}{n-2}})^i\right)$$

is uniformly bounded we obtain the statement of the Lemma. □

Now let u be a positive solution. The we obtain (6.17) for $p > 0$ as long as we avoid $p = 1/2$. In this case, if $1 > p_0 > 0$

$$\|u\|_{L^\infty(B_{1/4}(0))} \le c\|u\|_{L^{p_0}(B_{1/2}(0))} \tag{6.18}$$

and, with negative exponents (how we have to add ε to get a Lipschitz function)

$$\|u^{-1}\|_{L^\infty(B_{1/4}(0))} \le c\|u^{-1}\|_{L^{p_0}(B_{1/2}(0))}. \tag{6.19}$$

Suppose we knew that there exists p_0 so that

$$\|u\|_{L^{p_0}(B_{1/2}(0))}\|u^{-1}\|_{L^{p_0}(B_{1/2}(0))} \le C \tag{6.20}$$

Then

$$\begin{aligned}\sup_{x\in B_{1/4}} u(x)dx \le & c_1 \int_{B_{1/2}} u^{p_0}dx)^{1/p_0}dx \\ \le & c_2(\int_{B_{1/2}} u^{-p_0}dx)^{-\frac{1}{p_0}} \\ \le & c_3 \inf_{x\in B_{1/4}} u(x)\end{aligned}$$

which implies the claim by a covering argument.

We claim that $v = \eta(\ln u - \ln u_B) \in BMO$ and

$$\|v\|_{BMO} \le c(n)$$

Then, by the inequality of John-Nirenberg, Theorem 6.14,

$$\int_B e^{\delta|v-v_B|}dx \le c(n)$$

and hence,

$$\int_B u^\delta dx \int u^{-\delta}dx = \int_B e^{\delta(v-v_B)}dx \int_B e^{-\delta(v-v_B)}dx \le c(n)^2.$$

and this implies the previous statement.

We repeat the calculation above for $p = 0$, and $\tilde{\eta}$ as above, but related to a ball $B_r(x_0) \subset B_1(0)$,

$$\begin{aligned}0 = & \int_{B_1(0)} a^{ij}\partial_i u \partial_j(\tilde{\eta}u^{-1})dx \\ = & \int_{B_1(0)} a^{ij}\partial_i u(\partial_j\tilde{\eta})udx - \int a^{ij}\tilde{\eta}\partial_i u\partial_j uu^{-2}dx \\ = & \int_{B_1(0)} a^{ij}\partial_i(\ln u)\partial_j\tilde{\eta}dx - \int \tilde{\eta}a^{ij}\partial_i(\ln u)\partial_j(\ln u)dx\end{aligned}$$

$$\|\nabla \ln(u)\|_{H^1(B_{5/6}(0))} \le c$$

and hence, by Poincaré's inequality

$$\int_{B_{3/4}} |\ln(u) - \ln(u)_{B_{3/4}}|dx \le c.$$

We obtain for all balls contained in $B_{3/4}$

$$\|\nabla \ln u\|_{L^2(B_r(x))} \le cr^{-2}$$

hence (with Lemma 6.31 and more arguments, sorry, I'll complete that later)

$$\|\eta(\ln u - (\ln u)_B)\|_{BMO} \le c$$

□

THEOREM 6.28. *$p > n$ and $q > n/2$. Then there exists $s > 0$ so that the following is true. Suppose that $F^i \in L^p(B)$, $f \in L^q(B)$, $u \in H^1$,*

$$1 - \frac{n}{p} = 2 - \frac{n}{q} = s$$

$$\sum_{i,j=1}^n \partial_i a^{ij}\partial_j u = \partial_i F^i + g \qquad \text{in } B$$

Then

$$\sup_{x,y\in\frac12 B,x\neq y}\frac{|u(x)-u(y)|}{|x-y|^s}\le c\Big[R^{-\frac n2-s}\|u\|_{L^2(B)}+R^{1-\frac np-s}\|F\|_{L^p}+R^{2-\frac nq-s}\|f\|_{L^q}\Big]$$

PROOF. We prove this statement first for $F=g=0$ and $B=B_1(0)$. Let

$$\omega_k=\sup_{B_{2^{-k}}}u-\inf_{B_{2^{-k}}}u.$$

We claim that there exists $\gamma<1$ depending only on the quantities of the Harnack inequality so that

$$\omega_{k+1}<\gamma\omega_k. \tag{6.21}$$

Indeed, by the Harnack inequality applied to

$$v_k=u-\inf_{B_{2^{-k}}}u,w_k=\sup_{B_{2^{-k}}}u-u$$

$$\sup_{B_{2^{-k-1}}}u\le\inf_{B_{2^{-k}}}u+C(\inf_{B_{2^{-k-1}}}u-\inf_{B_{2^{-k}}}u)$$

and

$$\inf_{B_{2^{-k-1}}}u\ge\sup_{B_{2^{-k}}}u-C(\sup_{B_{2^{-k-1}}}u-\sup_{B_{2^{-k}}}u)$$

hence

$$(1+C)\omega_{k+1}\le(C-1)\omega_k,$$

which implies (6.21) with $\gamma=\frac{C-1}{C+1}<1$. Then, if $2^{-k-1}|x|\le2^{-k}$

$$|u(0)-u(x)|\le\omega_k\le\gamma^k\omega_0\le\gamma^{-\ln|x|}\|u\|_{L^2(B_10)}$$

and

$$\gamma^{-\ln|x|}=e^{-\ln\gamma\ln|x|}=|x|^{-\ln|\gamma|}.$$

□

Now we consider $U=\mathbb{R}^n$.

THEOREM 6.29. *Let* $n\ge3$ *There is a Green's function on* $\mathbb{R}^n$ *which satisfies*

$$|g(x,y)|\le c|x-y|^{2-n},|g(x,y)-g(\tilde x,y)|\le\frac{|x-\tilde x|^s}{(|x-y|+|\tilde x-y|)^{2+s-n}}$$

PROOF. Let F and f be supported in $B_1(0)$. By Lax Milgram (with H the space of functions in $L^{\frac{2n}{n-2}}$ with derivatives in L^2, equipped with the norm $\int|Du|^2dx$) there is a unique solution $u\in H^1_{loc}$ with $Du\in L^2$ and $u\in L^{\frac{2n}{n-2}}$ and

$$\|u\|_{L^{\frac{2n}{n-2}}}+\|Du\|_{L^2}\le c\Big[\|F\|_{L^2(B_1(0))}+\|g\|_{L^{\frac{2n}{n+2}}(B_1(0))}\Big].$$

By the previous Hölder estimate, if $|x_0|=3$,

$$\|u\|_{C^s(B_1(x_0))}\le c\Big[\|F\|_{L^2(B_1(0))}+\|g\|_{L^{\frac{2n}{n+2}}(B_1(0))}\Big]$$

We fix x and consider

$$L^{\frac{2n}{n+2}}(B_1(0))\ni f\to u^x.$$

By duality there exists a unique $g^x(y)$ with

$$\sup_x \|g^x(y)\|_{L^{\frac{2n}{n-2}}} \le C$$

and

$$u(x) = \int g^x(y) f(y) dy.$$

Clearly

$$\langle \partial_i a^{ij} \partial_j u, v \rangle = \langle \partial_i a^{ji} \partial_j v, u \rangle$$

Let $T : f \to u$. Then

$$\langle Tf, h \rangle = \langle f, Th \rangle$$

This operator is self adjoint. Hence $g^x(y) = g^y(x)$ and, repeating the previous argument we obtain the assertion.

□

Now we complete the prove of Theorem 6.28 by using the kernel estimates. Let $g^x(y)$ be the Green's function. We claim that

$$|\int_{B_1} (g^x(y) - g^{\tilde{x}}(y)) f(y) dy| \le c|x - \tilde{x}|^s \|f\|_{L^p(\mathbb{R}^n)}$$

By rescaling and translating we may assume that $\tilde{x} = 0$ and $|x| = 1$. We decompose $f = f_1 + f_2$ with $f_1 = \chi_{B_2(0)} f$. Then

$$\left| \int_{B_2(0)} g^0(y) f(y) dy \right| \le \|f\|_{L^p} \|g^0(.)\|_{L^{p'}(B_2(0))}$$

and the same holds for x and

$$\left| \int_{B_2(0)} (g^x(y) - g^0(y)) f(y) dy \right| \le \|f\|_{L^p} \|g^x - g^0\|_{L^{p'}(\mathbb{R}^n \backslash B_2(0))}.$$

Since $p' < \frac{n}{n-2}$ and by Theorem 6.29 $g(x,.) \in L_w^{\frac{n}{n-2}}$

$$\|g^0(.)\|_{L^{p'}(B_2(0))} \le c(n).$$

Similarly, again by Theorem 6.29

$$\|g^0 - g^x\|_{L_w^r(\mathbb{R}^n \backslash B_2(0))} \le c(n)$$

for

$$\frac{1}{r} = \frac{n-2}{n} - \frac{s}{n}.$$

This allows to bound the second term, provided we choose s in the theorem smaller than for the estimate of the solution to the homogeneous problem.

We proceed similarly with the term ∇F, for which we need

$$\|\nabla g^0\|_{L^{q'}(B_2(0))} \le c$$

and

$$\|\nabla (g^0 - g^x)\|_{L^{q'}(\mathbb{R}^n \backslash B_2(0))} \le c.$$

This follows by Cacciopoli's inequality

$$\begin{aligned}\|\nabla g^0\|_{L^{q'}(B_{2R}\setminus B_R)} \le & R^{\frac{n}{q'}-\frac{n}{2}}\|\nabla g^0\|_{L^2(B_{2R}\setminus B_R)} \\ \le & cR^{\frac{n}{q'}-\frac{n}{2}-1}\|g_0\|_{L^2(B_{4R}\setminus B_{R/2})} \\ \le & cR^{\frac{n}{q'}-(n-1)} \\ \le & cR^{1-\frac{n}{q}}\end{aligned}$$

which is summable over dyadic radii provided $q > n$. Similarly, if $R > 3$,

$$\|\nabla(g^0 - g^x)\|_{L^{q'}(B_{2R}\setminus B_R)} \le cR^{1-s-\frac{n}{q}}$$

which is clearly summable provided q is sufficiently large.

7. Pointwise estimates and perturbations of elliptic equations

We obtain an alternative argument for the boundedness of Calderón zygmund operators with kernels satisfying

$$|K(x,y)| \le c|x-y|^{-n}, |D_{x,y}K(x,y)| \le c|x-y|^{-n-1}$$

THEOREM 6.30. *Let $r > 1$. Then*

$$Tf^\sharp(x) \le c(M|f|^r)^{1/r}(x).$$

PROOF. We fix a ball $B_1(0)$ and decompose $f = f_1 + f_2$ with $f_1 = \chi_{B_3(0)} f$. Then

$$\|Tf_1\|_{L^r} \le c\|f_1\|_{L^r} \le c(M|f|^r(x))^{1/r}$$

and, for $x, \tilde{x} \in B_1(0)$,

$$|Tf_2(x) - Tf_2(\tilde{x})| \le cMf_2(x).$$

□

Thus for $p > r$

$$\|Tf\|_{L^p} \le \|(Tf)^\sharp\|_{L^p} \le c\|(M|f|^r)^{1/r}\|_{L^p} \le c\|f\|_{L^p}.$$

This proof uses the Calderón-Zygmund estimate. Instead we could set $r = p_0$, get bounded for $p > p_0$, and apply duality.

LEMMA 6.31. *Let $f \in BMO$. Then*

$$|f_{B_{r_1}(x_1)} - f_{B_{r_2}(x_2)}| \le c(|\ln r_1/r_2| + \ln\frac{|x_1-x_2| + r_1 + r_2}{r_1 + r_2})\|f\|_{BMO}$$

PROOF. We prove this statement in several steps. We have

$$\begin{aligned}|f_{B_R(y)} - f_{B_r(x)}| \le & \left| |B_r(x)|^{-1}\int_{B_r(x)} f - f_{B_R(y)}dx\right| \\ \le & (R/r)^n|B_R(y)|^{-1}\int |f - f_{B_R(y)}|dx \le (R/r)^n f^\sharp(y)\end{aligned}$$

if $B_r(x) \subset B_R(x)$. If $\max\{|x_1 - x_2|, r_1\} < r_2 < 2r_1$ then we obtain immediately the assertion. Moreover, by applying the argument k times

$$|f_{B_{2^k r}(x)} - f_{B_r(x)}| \le 2^n k f^\sharp(x)$$

We choose

$$k = \max\{|\ln r_2/r_1|, \ln|x_1 - x_2|/(r_1 + r_2)\} + 2$$

and $R = 2^k \min\{r_1, r_2\}$. Then

$$\begin{aligned} |f_{B_{r_1}(x_1)} - f_{B_{r_2}(x_2)}| \leq & |f_{B_{r_1}(x_1)} - f_{B_R(x_1)}| + |f_{B_{r_2}(x_2)} - f_{B_R(x_2)}| \\ & + |f_{B_R(x_1)} - f_{B_R(x_2)}| \\ \leq & (2k+1)2^n f^\sharp(x_1) \end{aligned}$$

□

Let $a^{ij} \in L^\infty(\mathbb{R}^n)$ and

$$a^{ij}\xi_i\xi_j \geq \kappa|\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and almost every x. We consider u with second derivatives in L^p and and

$$\sum_{i,j=1}^n a^{ij}\partial^2_{ij}u = f.$$

THEOREM 6.32. *Let $p \in (1,\infty)$, $a^{ij} \in L^\infty$ and $\kappa > 0$. There exists $\varepsilon > 0$ such that if*

$$\|a^{ij}\|_{BMO} < \varepsilon$$

then

$$\|D^2u\|_{L^p} \leq c\|f\|_{L^p}.$$

REMARK 6.33. *This theorem provides an important and strong theme in linear and nonlinear partial differential equations: Often not pointwise conditions on coefficients are important, but uniform conditions on every scale. This type of result goes back to Caffarelli and Xabre [1] in 1995 by different techniques.*

PROOF. We claim that with $r > 1$

$$D^2u^\sharp \leq c(M|f|^r)^{1/r} + c\|a\|_{BMO}(M|D^2u|^r)^{1/r}. \tag{6.22}$$

Then

$$\|D^2u\|_{L^p} \leq c\|f\|_{L^p} + c\|a^{ij}\|_{BMO}\|D^2u\|_{L^p}.$$

The claim follows if $c\|a^{ij}\|_{BMO} < \frac{1}{2}$.

As for the proof of Theorem 6.30 we fix a ball $B = B_1(0)$. To simplify the notation we assume that $a^{ij}_B = \delta^{ij}$.

We rewrite the equation as

$$\Delta u = f + (\delta^{ij} - a^{ij})\partial^{ij}u$$

and, redoing the argument above

$$\begin{aligned} \|\partial^{ij}u - \partial^{ij}u_B\|_{L^1(B)} \leq & c\|f\|_{L^r(2B)} + Mf(x) \\ & + c\|(\delta^{ij} - a^{ij})\partial^{ij}u\|_{L^{\frac{1+r}{2}}(2B)} \\ & + c\int_{|y|>2} |y|^{-n-1}|(\delta^{ij} - a^{ij})\partial^{ij}u|dy. \end{aligned}$$

Then,by Hölder inequality, with $\frac{1}{q} = \frac{2}{1+r} - \frac{1}{r}$ and the triangle inequality

$$\begin{aligned}\|(\delta^{ij} - a^{ij})\partial^{ij} u\|_{L^{\frac{1+r}{2}}(2B)} \leq & \Big[\|a^{ij}_{2B} - a^{ij})\|_{L^q} + |a^{ij}_{2B} - a^{ij}_B|\Big](M|D^2u|^r(0))^{1/r} \\ \leq & c\|a\|_{BMO}(M|D^2u|^r(0))^{1/r}\end{aligned}$$

where we also used Lemma 6.31 and the proof of Theorem 6.14.

Similarly

$$\begin{aligned}&\left|\int_{B_{2^k}(0)\setminus B_{2^{k-1}(0)}} |y|^{-n-1}|a^{ij} - \delta^{ij}||D^2u|dy\right| \\ &\qquad \leq 2^{-k}\left(2^{-\frac{nk}{r'}}\|a^{ij} - a^{ij}_{B_{2^k}}\|_{L^{r'}(B_{2^k}(0))} + |a^{ij}_{B_{2^k}(0)} - a^{ij}_B|\right) \\ &\qquad\qquad \times \left(2^{-\frac{nk}{r}}\|D^2u\|_{L^r(B_{2^k})}\right) \\ &\qquad \leq 2^{-k}(1+k)\|a\|_{BMO}(M|D^2u|^r(0))^{1/r}.\end{aligned}$$

This implies (6.22). □

CHAPTER 7

Littlewood-Paley theory and square functions

1. The range $1 < p < \infty$

We fix a function $\phi \in \mathcal{S}$ with $\int \phi dx = 0$. This conditions can be relaxed in the sequel. For $f \in L^p$ we define

$$(Tf)(t,x) = f * \phi_t(x) =: F(t,x) \in L^2(\mathbb{R}^n \times \mathbb{R}_+, dxdt/t)$$

Then the formal adjoint operator is

$$T^* F(x) = \int \phi_t(y-x) F(t,y) \frac{dt}{t}\, dy.$$

Then T^*T is the Fourier multiplier given by

$$m(\xi) = \int_0^\infty \overline{\phi(t\xi)} \phi(t\xi) \frac{dt}{t}$$

which is homogeneous of degree 0 and smooth.

We consider T as an operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n, L^2(dt/t))$. Its kernel is given by the convolution kernel

$$K(x-y) = (t \to t^{-n}\phi((x-y)/t)) \in L^2(dt/t)$$

and

$$\begin{aligned} |K(z)| = & \left(\int_0^\infty t^{-1-2n} |\phi(z/t)|^2 dt \right)^{1/2} \\ = & |z|^{-n} \omega(\frac{z}{|z|}) \end{aligned}$$

where

$$\omega(\nu) = \left(\int_0^\infty t^{-1-2n} |\phi(\nu/t)|^2 dt \right)^{1/2}$$

is bounded since

$$|\phi(\nu/t)| \le c \min\{t^{n+\varepsilon}, 1\}$$

Similarly

$$\begin{aligned} |\partial_{x_i} K(z)| = & \left(\int_0^\infty t^{-3-2n} |(\partial_i \phi)(z/t)|^2 dt \right)^{1/2} \\ = & |z|^{-n-1} \omega_i(\frac{z}{|z|}) \end{aligned}$$

The same type of estimate holds for T^*.

The Calderón-Zygmund Theorem 5.4 applies to vector valued functions, see the Remark 5.7 and we obtain

Proposition 7.1. *Let $1 < p < \infty$,$A > 0$ and $\varepsilon > 0$. We assume that $\phi \in C(\mathbb{R}^n)$ satisfies*

$$|\phi(x)| \le A(1+|x|)^{-n-\varepsilon}, \qquad |D\phi(x)| \le A(1+|x|^{-n-1-\varepsilon}, \qquad \int \phi dx = 0. \tag{7.1}$$

Then there exists c depending on p, n and ε so that

$$\|Tf\|_{L^p(\mathbb{R}^n, L^2(\frac{dt}{t}))} \le c\frac{p^2}{p-1}(A+\|\omega\|_{L^\infty})\|f\|_{L^p(\mathbb{R}^n)}$$

and

$$\|T^*F\|_{L^p(\mathbb{R}^n} \le c\frac{p^2}{p-1}(A+\|\omega\|_{L^\infty})\|F\|_{L^p(\mathbb{R}^n;L^2(\frac{dt}{t}))}$$

Proof. We have seen that the operator satisfies the L^2 bound under the assumption. The kernel estimates also hold. Hence the assertion follows for $1 < p \le 2$ for T and T^*. Duality gives the full statement. □

Examples are

(1) The derivatives of the heat kernel, evaluated at $t = 1$:

$$\pi^{-n/2}(|x|^2 - \frac{n}{2})e^{-\frac{|x|^2}{2}}, \qquad \pi^{-n/2}x_j e^{-\frac{|x|^2}{2}}$$

(2) The derivatives of the Poisson kernel

$$c_n\frac{|x|^2-(n+1)}{(1+|x|^2)^{\frac{n+3}{2}}}, \qquad c_n\frac{x_j}{(1+|x|^2)^{\frac{n+2}{2}}}$$

Given ϕ we search ψ so that

$$T_\psi^* T_\phi f = f$$

where use the index ϕ and ψ with the obvious meaning. For that we need more angular regularity. Let $i = (i_1, i_2)$ be a pair of (nonequal) indices between 1 and n, and we define the angular derivatives

$$D_i = x_{i_1}\partial_{i_2} - x_{i_2}\partial_{i_1}.$$

Let α be the analogue of the multiindices for pairs of indices, and we denote - by an abuse of notation

$$D^\alpha$$

for the obvious product of angular derivatives. The angular derivative of the Fourier transform is the Fourier transform of angular derivatives.

Lemma 7.2. *Suppose that ϕ satisfies the conditions* (7.1) *and*

$$|D^\alpha \phi| \le c_\alpha |x|^{-n-\varepsilon} \tag{7.2}$$

for all such multiindices and assume and $\hat\phi$ does not vanish identically on any set $\{\lambda\nu : \lambda > 0\}$ with $|\nu| = 1$. Then there exists $\psi \in \mathcal{S}$ satisfying $\int \psi = 0$ and hence (7.1) *and*

$$\int_0^\infty \hat\phi(t\xi)\overline{\hat\psi(t\xi)}\frac{dt}{t} = 1$$

for all $\xi \neq 0$.

Remark 7.3. *In particular*

$$T_\psi^* T_\phi f = T_\psi^* T_\phi f = f$$

for all $f \in L^2$ *and hence for all* f *in* L^p.

Proof. Let $|\nu| = 1$. Then there exist $\varepsilon > 0$, r and R so that

$$\int_r^R |\hat\phi(\nu/t)| > \varepsilon$$

by continuity and compactness we can choose ε, r and R independent of ν. Given ν we fix a smooth compactely supported function ψ_ν on $[r/2, 2R]$ with

$$\int \hat\phi(t\nu)\psi_\nu(t)\frac{dt}{t} = 1$$

Then

$$\xi \to \int \hat\phi(t\xi/|\xi|)\overline{\psi_\nu(t)}\frac{dt}{t} = \rho_\nu(\xi)$$

is smooth due to condition (7.2). Locally we can divide by ρ_ν. We use a homogeneous partition η_k of unity on $\mathcal{S}^{n-1}$ to construct ψ by

$$\hat\psi(\xi) = \sum \frac{\eta_k(\xi)}{\rho_{\nu_k}(\xi)}\psi_{\nu_k}(|\xi|) \in \mathcal{S}.$$

supported in $B_{2R}\backslash B_{r/2}$. □

Definition 7.4 (Square function). *Let* ϕ *satisfy* (7.1) . *We define*

$$s_f(x) = \left(\int |f * \phi_t(x)|^2 \frac{dt}{t}\right)^{1/2}$$

$$s_f^*(x) = \left(\int_t t^{-n-1}\int_{|y|<t} |f * \phi_t(x+y)|^2 dydt\right)^{1/2}$$

and

$$g_f(x) = \left(\sum_k |f * \phi_{2^k}|^2(x)\right)^{1/2}.$$

Theorem 7.5. *Suppose that* $1 < p < \infty$. *Then*

$$\|s_f\|_{L^p} \le c\|f\|_{L^p},$$
$$\|s_f^*\|_{L^p} \le c\|f\|_{L^p}$$

and

$$\|g_f\|_{L^p} \le c\|f\|_{L^p}.$$

If ϕ *satisfies the assumption of Lemma 7.2 then*

$$\|f\|_{L^p} \le c\|s_f\|_{L^p},$$
$$\|f^*\|_{L^p} \le c\|s_f^*\|_{L^p}.$$

If ϕ *satisfies the assumptions of Lemma 7.2 with:*

For all $\xi \neq 0$ *there exists* k *so that*

$$\hat\phi(2^k\xi) \neq 0.$$

replacing the integral condition then

$$\|f\|_{L^p} \le c\|g_f\|_{L^p}.$$

PROOF. The claims on s_f follow from Proposition 7.1 and Lemma 7.2. For $s^*(f)$ we observe that

$$\begin{aligned}\|s^*_f\|^2_{L^2} =& \int_{\mathbb{R}^n}\int_0^\infty t^{-n-1}\int_{|y|\le t}|f*\phi_t(x-y)|^2 dydtdx\\ =&\int_{|y|\le 1}\int_{\mathbb{R}^n}\int_0^t |f*(\phi(.+y))_t)(x)|^2\frac{dt}{t}dxdy\\ =&|B_1|\int_{\mathbb{R}^n}|\hat f|^2\int_0^\infty|\hat\phi(t\xi)|^2\frac{dt}{t}d\xi\end{aligned}$$

and we obtain the L^2 bound. It also follows that with ψ as in Lemma 7.2 - assuming that the assumptions are satisfies - ψ defines a left inverse to the vector values operator.

Similarly we obtain the kernel bound for $T_{\phi(.+y)}$ for all $|y| \le 1$, and hence also when we integrate y with respect to the unit balls.

Replacing the t integration by a summation for g_f does not require substantial changes. Only the analogue of Lemma 7.2 needs some consideration. We want to find ψ so that

$$\sum_{k=-\infty}^{\infty}\hat\phi(2^k\xi)\overline{\hat\psi}(2^k\xi) = 1$$

for $\xi \ne 0$. This is done as in Lemma 7.2. □

2. Square functions, tents and Carleson measures

Theorem 6.10 applies here and gives

$$\|\int_0^t|\phi_t*f|^2\frac{dt}{t}\|_{L^1(\mathbb{R}^n)} \le c\|f\|_{\mathcal{H}^1}.$$

We also have

LEMMA 7.6. *Let ϕ satisfy* (7.1). *Then*

$$\sup_{x,R}|B_R|^{-1}\int_{B_R(x)}\int_0^R|f*\phi_t|^2\frac{dt}{t}dy \le c\|u\|^2_{BMO}$$

PROOF. As usual is suffices to prove the bound for $B = B_1(0)$. We may assume that $f_{2B} = 0$, since we may add a constant. We write

$$f = f_1 + f_2$$

with $f_1 = f\chi_{2B}$. Then $f_1 \in L^2$ and $T_\phi f_1 \in L^2(\mathbb{R}^n\times(0,\infty), dx\frac{dt}{t})$. It remains to consider f_2. We claim that

$$|T\phi f_2(t,x)| = |f_2*\phi_t(x)| \le ct\|f\|_{BMO}$$

if $|x| + t \le 1$. This follows from Lemma 6.31,

$$\int_{|y|\ge 1}|y|^1|\phi(y)|dy \le cT$$

and scaling. The inequality is a consequence of the bounds on ϕ. □

It will be useful to consider more general functions F. We define

$$s^*(F)(x) = \left(\int_{|y|\le t} t^{-n-1} |F(t,x+y)|^2 dydt \right)^{1/2}$$

and the analogues construction based on tends instead of cones

$$C(F)(x) = \sup_R \left(\int_{|y|+t<R} t^{-1} |F(t,x+y)|^2 dydt \right)^2 .$$

We call $|F|^2$ a Carleson measure if C_F is bounded.

The following duality statement holds.

Proposition 7.7.

$$\left| \int_{\mathbb{R}^n} \int_0^\infty F(t,x)G(t,x) \frac{dt}{t} dx \right| \le c \int s_F^* C_G dx.$$

Proof. We define $\tau(x)$ for a large constant A by

$$\tau(x) = \sup \Big\{ \tau > 0 : \int_0^\tau \int_{|y|<t} t^{-n-1} G^2(t,x-y) dtdy \le AC_G(x) \Big\}.$$

We claim that there exists A_0 and $c > 0$ so that if $A \ge A_0$ then for all balls $B = B_r(x)$

$$|\{x \in B : \tau(x) \ge r\}| \ge c|B_r(x)|. \tag{7.3}$$

Then, for any nonnegative function H

$$\int_{\mathbb{R}^n} \int_0^\infty H(y,t) t^n dydt \le c^{-1} \int_{\mathbb{R}^n} \int_0^{\tau(x)} \int_{|y|\le t} H(y,t) dydtdx.$$

We take $H = FGt^{-n-1}$ and get

$$\begin{aligned} \int_{\mathbb{R}^n} \int_0^\infty F(t,x)G(t,x)\frac{dt}{t} dx \le & c^{-1} \int s^*(F)(x) \times \\ & \int_0^{\tau(x)} \int_{|y|\le\tau} t^{-n-1} G^2(t,x-y) dydt \\ \le & \frac{A}{c} \int_{\mathbb{R}^n} s_F^* C_G dx \end{aligned}$$

by the definition of $\tau(x)$.

It remains to prove (7.3). Let $B = B_r(x_0)$ be a ball, $y \in B$, and $T = \{(t,x) : |x-y| + t < 3r\}$ the tent over the ball of 3 times the radius around y. Thus, by an application of Fubini, similar to the treatment of $s*$,

$$\int_B \int_0^r \int_{|y|<t} G^2(t,x-y) dydt \le |B_1(0)| \int_T |G(t,y)|^2 dy \frac{dt}{t}$$

Hence

$$\int_B \int_0^r \int_{|y|<t} G^2(t,x-y) dydt \le c_n \inf_{y\in B} \int_T |G|^2 dy \frac{dt}{t} \le c_n \inf_{y \in B} C_G(y).$$

This implies (7.3) if we choose $A_0 > c_n$. □

THEOREM 7.8. *Suppose that ϕ satisfies the assumptions of Lemma 7.2. Then*

$$\|f\|_{BMO} \leq c\|C_{Tf}\|_{L^\infty}$$

and

$$\|f\|_{\mathcal{H}^1} \leq c\|s^*_{Tf}\|_{L^1}$$

PROOF. Let ψ be as in Lemma (7.2). Then

$$\int f\overline{g}dx = \int T_\phi f\overline{T_\psi g}\frac{dt}{t}dx$$

By duality and Proposition 7.7

$$\begin{aligned}
\|f\|_{BMO} \leq & C\sup\left\{\int fgdx : \|g\|_{\mathcal{H}^1} \leq 1\right\} \\
= & C\sup\left\{\int TfTg\frac{dt}{t}dx : \|g\|_{\mathcal{H}^1} \leq 1\right\} \\
= & C\sup\left\{\int C_{Tf}s^*_{Tg}dx : \|g\|_{\mathcal{H}^1} \leq 1\right\} \\
\leq & C\|C_{Tf}\|_{L^\infty}\sup\{\|s^*_{Tg}\|_{L^1} : \|g\|_{\mathcal{H}^1} \leq 1\} \\
& \|f\|_{BMO} \leq c\|C_{Tf}\|_{L^\infty}
\end{aligned}$$

and similarly

$$\|g\|_{\mathcal{H}^1} \leq c\|S_{Tg}\|_{L^1}.$$

□

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