CHAPTER 6

Hardy and BMO

1. More general maximal functions and the Hardy space $\mathcal{H}^p$

We fix a measurable function ϕ for which there is a radial and radially decreasing majorant ϕ^*, $|\phi| \le \phi^*$. Then we have seen that

$$|\phi * f(x)| \le cMf(x).$$

We define $\phi_t(x) = t^{-n}\phi(x/t)$.

For $N \in \mathbb{N}$ we define the norm

$$\|f\|_N = \sup_{|\alpha|+|\beta|\le N} \sup_x |x^\alpha \partial^\beta f|$$

and the set of functions

$$\mathcal{F}_N = \{\phi : \|\Phi\|_N \le 1\}.$$

Definition 6.1. *We define*

$$M_\phi f(x) = \sup_t |f * \phi_t(x)|, \tag{6.1}$$

the nontangential version

$$M^*_\phi f(x) = \sup_t \sup_{|y|\le t} |f * \phi_t(x+y)| \tag{6.2}$$

and the 'grand' maximal function

$$M_N f(x) = \sup_{\phi:\|\phi\|_N\le 1} \sup_t |f * \phi_t(x)|. \tag{6.3}$$

It is important in the following theorem that we allow $p \le 1$.

Theorem 6.2. *Let f be a tempered distribution and $0 < p \le \infty$. Then the following conditions are equivalent*

1. *There exists $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \phi = 1$ so that $M_\phi f \in L^p$.*
2. *There exist seminorms N so that $M_N \phi \in L^p$*
3. *$M^*_{e^{-\pi|x|^2}} f \in L^p$.*

We define the *real* Hardy space $\mathcal{H}^p$ as the set of all functions for which the equivalent conditions of the Theorem hold.

If $p > 1$ then any maximal function of f majorizes a multiple of f. The second and the third are bounded by the standard Hardy-Littlewood maximal function and hence $\mathcal{H}^p = L^p$ in that case.

For $p = 1$ the same argument shows that $\mathcal{H}^1 \subset L^1$. The spaces L^p for $p < 1$ are defined in the obvious fashion. They are not Banach spaces, and they do not imbed into the space of distributions.

There are typical elements of $\mathcal{H}^p$ called atoms.

Definition 6.3 (Atoms). *Let $0 < p \le 1$ A p atom is a bounded function a for which there is a ball $B = B_r(x_0)$ so that*

$$\operatorname{supp} a \subset B$$
$$|a| \le |B|^{-1/p}$$
$$\int x^\alpha a dx = 0$$

for all multiindices α with

$$|\alpha| \le n(\frac{1}{p} - 1)$$

Lemma 6.4. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and let a be a atom with the ball $B_r(x_0)$. Then there exist ε and c so that*

$$M_\phi a \le C|B_r(x_0)|^{-1/p}(1 + \frac{|x - x_0|}{r})^{-\frac{n}{p} - \varepsilon(n,p)}$$

In particular

$$\int |M_\phi a|^p dx \le c(n,p).$$

Proof. Exercise □

We introduce a modified nontangential maximal function for $a \ge 1$

$$M^a f(x) = \sup_t \sup_{|y| \le at} |f * \phi_t(x - y)|$$

Lemma 6.5.

$$\|M^a f\|_{L^p} \le c a^{n/p} \|f\|_{L^p}$$

Proof. The claim follows from

$$|\{x : M^a f > \lambda\}| \le c a^n |\{x : M_\phi^* f > \lambda\}| \tag{6.4}$$

by integration.

Let $O = \{x : M_\phi^* f > \lambda\}$. Suppose that $M^a f(x) > \lambda$. Then there exist $(\tilde{x}, \tilde{t})$ with $f * \phi_{\tilde{t}}(\tilde{x}) > \lambda$ and $|x - \tilde{x}| \le at$. Then $B_t(\tilde{x}) \subset O$ and hence

$$\frac{|O \cap B_{at}(x)|}{|B_{at}(x)|} > a^{-n}.$$

Let $A = \mathbb{R}^n \backslash O$ and

$$A^* = \{x \in A : \frac{|O \cap B_r(x)|}{|B_r(x)|} < a^{-n} \text{ for some } r\}$$

Then

$$|\mathbb{R}^n \backslash A^*| \le (3a)^n |\mathbb{R}^n \backslash A| \tag{6.5}$$

implies (6.4).

To prove (6.5) we turn to an argument in measure theory. Suppose that $A \subset \mathbb{R}^n$ is a closed set and let $0 < \gamma < 1$ ($\gamma = 1 - a^{-n}$) . Let $A^* \subset A$ be the set of all points x so that

$$\frac{|A \cap B|}{|B|} \le \gamma$$

for some ball B containing x. Then

$$\mathbb{R}^n \backslash A^* = \left\{x : M(\chi_{\mathbb{R}^n \backslash A}) > 1 - \gamma\right\}$$

and by the estimate for the maximal function

$$|\mathbb{R}^n \backslash A^*| \le \frac{3^n}{1-\gamma}|\mathbb{R}^n\backslash A|$$

□

For the proof of Theorem 6.2 we have to study the effect of changing the function is the definition of the maximal function. This is the easier part of the proof.

LEMMA 6.6. *Let $\phi, \psi \in \mathcal{S}$ with $\int \phi = 1$. For all $M > 0$ There exists a sequence $\eta^{(k)} \in \mathcal{S}$ such that for all N*

$$\|\eta^{(k)}\|_N \le c_{N,M} 2^{-kM}$$

and

$$\psi = \sum \eta^{(k)} * \Phi_{2^{-k}}$$

PROOF. We fix $\rho \in C_0^\infty(B_2(0))$, identically 1 on B_1 and define

$$\rho_k(\xi) = \rho(2^{-k}\xi) - \rho(2^{1-k}\xi)$$

for $k \ge 1$ and $\rho_0 = \rho$. Then

$$\hat{\psi} = \sum_{k=0}^{\infty} \rho_k \hat{\psi}.$$

Now $1 = \int \phi dx = \hat{\phi}(0)$. Without loss of generality we assume $|\hat{\phi}(\xi)| \ge \frac{1}{2}$ for $|\xi| \le 2$. Then

$$\hat{\psi}(\xi) = \sum_{k=0}^{\infty} \frac{\rho_k(\xi)}{\hat{\phi}(2^{-k}\xi)} \hat{\Psi}(\xi)\hat{\phi}(2^{-k}\xi) = \hat{\eta}^{(k)}\hat{\phi}(2^{-k}\xi)$$

Now $\hat{\Psi}$ is a Schwartz function which leads to the claimed decay. □

The proof gives actually a stronger statement: Given M there exists N so that the claim holds for $\|\psi\|_N < \infty$.

We turn to the proof of the theorem.

PROOF. Let $\Phi \in \mathcal{S}$ with $\int \Phi = 1$. We claim that there is are constants c and N so that

$$\|\mathcal{M}_N f\|_{L^p} \le c \|M_\Phi^* f\|_{L^p} \tag{6.6}$$

and

$$\|M_\Phi^* f\|_{L^p} \le c\|M_\Phi f\|_{L^p} \tag{6.7}$$

These two estimates imply all assertions of the theorem.

To proof the first inequality we choose $\psi \in \mathcal{S}$. The expansion gives

$$\begin{aligned} M_\psi f(x) = & \sup_t |f * \psi_t(X)| \le \sup_{t>0} \sum_{k=0}^{\infty} |f * \Phi_{2^{-k}t} * \eta_t^{(k)}(x)| \\ \le & \sup_t t^{-n} \sum_k \int |f * \Phi_{2^{-k}t}(x-y)||\eta^{(k)}(y/t)|dy \end{aligned}$$

$$\le \sup_t \sum_k \sup_y |f * \Phi_{2^{-k}t}(x-y)|(1+\frac{|y|}{2^{-k}t})^{-N} \int t^{-n}(1+\frac{|y|}{2^{-k}t})^N |\eta^{(k)}(y/t)|dy$$

with $N > n/p$ and

$$t^{-n}\int (1+\frac{|y|}{2^{-k}t})^N |\eta^{(k)}(y/t)|dy \le c2^{-k}$$

if (which is ensured by Lemma 6.6)

$$\|\eta^{(k)}\|_{N+n} \le c2^{-k(N+1)}.$$

We claim that

$$\| \sup_t \sup_y |f * \Phi_t(x-y)(1+\frac{|y|}{t})^{-N}\|_{L^p} \le c\|M^*_\phi f(x)\|_{L^p}. \tag{6.8}$$

Then

$$\sup_t \sup_y |f * \Phi_t(x-y)(1+\frac{|y|}{t})^{-N} \le \sup_{j=0,1\dots} 2^{-jN} M^{2^j} f(x)$$

and the assertion follows Lemma 6.5.

This implies (6.6).

To complete the proof we will prove

$$\|M^*_\phi f\|_{L^p} \le c\|f\|_{L^p}. \tag{6.9}$$

Let

$$F_\lambda = \{x : M_N f(x) \le \lambda M^*_\phi f(x)\}.$$

Since

$$\int_{\mathbb{R}^n\setminus F} |M^*_\Phi f|^p dx \le \lambda^{-p}\int_{\mathbb{R}^n\setminus F} |M_N f|^p dx \le c^p\lambda^{-p}\int |M^*_\Phi f|^p dx$$

we obtain

$$\int_{\mathbb{R}^n} |M^*_\Phi f|^p dx \le 2\int_F |M^*_\Phi f|^p dx$$

provided we choose $\lambda^p \ge 2c^p$.

We claim that on F and any $q > 0$

$$M^*_\Phi f(x) \le c[M|M_\Phi f|^q(x)]^{1/q}. \tag{6.10}$$

This implies the desired estimate via

$$\int_{\mathbb{R}^n} |M^*_\Phi f|^p dx \le 2\int_F |M^*_\Phi f|^p dx \le c_1\int [M|M_\Phi f|^q]^{p/q} dx \le c_2\int |M_\Phi f|^p dx$$

by the estimate for the Hardy-Littlewood maximal function. It remains to prove (6.10). Let

$$f(x,t) = f * \Phi_t(x), \qquad f^*(x) = M^*_\Phi f(x)$$

By definition, for any x there exists (y,t) with $|x-y| \le t$ so that

$$|f(y,t)| \ge f^*(x).$$

By the fundamental theorem of calculus

$$|f(x',t) - f(y,t)| \le rt \sup_{|z-y|<rt} |D_x f(z,t)|$$

for $|x'-y| \le rt$. However

$$\partial_{x_i} f(z,t) = \frac{1}{t} f * (\partial_i\Phi)_t(z)$$

hence

$$|f(x',t) - f(y,t)| \le cfM_N f(x) \le cr\lambda M^*_\Phi f(x) = c\lambda r f^*(x)$$

if $x \in F$. We take r so small that $c\lambda r \le \frac{1}{4}$ to achieve

$$|f(x',t)| \ge \frac{1}{4} f^*(x) \tag{6.11}$$

for $|x'-y| \le rt$. Thus

$$\begin{aligned} |M^*_\Phi f(x)|^q \le & \left(\frac{1+r}{r}\right)^n \frac{4^q}{|B_{(1+r)t}(x)|} \int_{B_{x,(1+r)t}} |f(x',t)|^q dx' \\ \le & cM[(M_\Phi f)^q](x). \end{aligned}$$

The second inequality follows from

$$|f(x',t)| \le M_\phi(x')$$

and the first from the lower bound (6.11).

There is a last tricky part: We severely used that $\|M^*_\phi f(x)\|_{L^p} < \infty$. To deal with that we repeat the arguments with

$$M^{\varepsilon,L}_\Phi f(x)) = \sup_{|x-y|<t<\varepsilon^{-1}} |f * \Phi_t(y)| \frac{t^L}{(\epsilon + t + \epsilon|y|)^L}$$

instead of M^*. If f is a tempered distribution we choose L large and ε small so that $\|M^{\varepsilon,L}_\Phi f\|_{L^p} < \infty$

Then we introduce the factor

$$\frac{t^L(\varepsilon + 2^{-k}t + \varepsilon|x-y|)^L}{\varepsilon + t + \varepsilon|x|)^{-L}(2^{-k}t)^{-L}(1+\frac{2^k|y|}{t})^N} \le c2^{kL}(1+\frac{|y|}{t})^L(1+\frac{2^k|y|}{t})^N$$

as suitable points. We complete the proof as above.

□

2. The atomic decomposition

The key part of the proof is a refined Calderón-Zygmund decomposition.

We recall that we can write any nonempty set $U \subset \mathbb{R}^n$, $U \ne \mathbb{R}^n$ as the union of dyadic cubes

$$Q_{kl} = 2^l([0,1)^n + k)$$

such that the length of the edge is at least the distance to the complement, and at most n times the distance. We fix two numbers $1 < a < b < 1+1/(4n)$ and denote $\tilde{Q} = aQ$, $Q^* = bQ$ where aQ resp bQ denotes the cube Q scaled by a with center the center of the cube.

Proposition 6.7. *Let $f \in L^1_{loc}$ with $M_{e^{-2\pi|x|^2}} f \in L^1$ and $\lambda > 0$. Then there is a decomposition*

$$f = g + b, b = \sum b_k$$

and a collection of dyadic cubes Q_k so that

(1) $|g| \le c(n)\lambda$
(2) $\operatorname{supp} b_k \subset Q^*_k$ *and* $\int b_k dx = 0$

(3) *The Q_k are disjoint and*

$$\bigcup Q_k = \{x : M_N f > \lambda\}$$

PROOF. We fix $\zeta \in C_0^\infty(((0,1)^n)^*)$, identically one on $[0,1]^n$. For $k \in \mathbb{Z}^n$ and $l \in \mathbb{Z}$ we define

$$\zeta_{kl} = \zeta(2^{-l}x - k)$$

which is supported in Q^*_{kl} and identically 1 in $\tilde{Q}_{kl}$.

Let $O = \{x : M_N f(x) > \lambda\}$, let Q_{k_j,l_j} be a Whitney decomposition (with disjoint cubes, and edge lengths comparable to the distance to the complement), and

$$\eta_j = \frac{\zeta_{k_j,l_j}}{\sum_i \zeta_{k_i,l_i}}$$

a partition of unity. Then, if l_j is the edge length,

$$|\partial^\alpha \eta_j| \le c 2^{-l_j|\alpha|}.$$

We define

$$b_j = (f - c_j)\eta_j, \qquad c_j = \frac{\int f\eta_j dx}{\int \eta_j}$$

Then, by the definition of Q_j there exist r and $x \in \mathbb{R}^n \backslash O$ such that

$$Q_j \subset B_r(x_0)$$

and $r \le c(n)|Q_j|^{1/n}$. But then

$$\|\eta_j(\frac{x - x_0}{r})\|_N \le c(n)$$

and

$$\left|\int \eta_j f dx\right| \le c(n) r^n M_N f(x_0) \le c(n) r^n \lambda.$$

Thus

$$|c_j| \le c(n)\lambda.$$

Now

$$|g| \le c M_N f(x) \le c\lambda$$

for $X \notin O$. Together this gives the bound on g. $\square$

THEOREM 6.8. *Suppose that $0 < p \le 1$ and $f \in \mathcal{H}^p$. Then there exists a sequence of p atoms a_j and a summable sequence λ_j so that*

$$f = \sum \lambda_j a_j$$

and

$$\sum |\lambda_j|^p \le c(n,p) \|M_{e^{-\pi|x|^2}} f\|^p_{L^p}.$$

REMARK 6.9. *Since*

$$|M_\phi \sum_j \lambda_j a_j|^p \le \sum_j |\lambda_j|^p |M_\phi a_j|$$

$$= \sum_j |\lambda_j| |M_\phi a_j|^p$$

any such sum is bounded in $\mathcal{H}^p$. The sum

$$\sum_{\lambda_j} a_j$$

converges in the space of tempered distributions.

Proof. We only consider $p = 1$. We have seen that $\mathcal{H}^1 \subset L^1$. Let $f \in \mathcal{H}^1$. It is integrable. For each integer l we apply the Calderón-Zygmund decomposition at level 2^l and we write $f = g^l + b^l$, $b^l = \sum_j b^l_j$.

We claim that

$$g^l \to f$$

in $\mathcal{H}^1$ for $l \to \infty$, or, equvalently, $\|b^l\|_{\mathcal{H}^1} \to 0$ as $l \to \infty$. This follows from

$$\begin{aligned} \|b^l\|_{\mathcal{H}^1} &\sim \int M_{e^{-\pi|x|^2}} b^j dx \\ &\le \sum_j \int M_{e^{-\pi|x|^2}} b^j_k dx \\ &\le \int_{\bigcup Q^l_j} (M_N f) dx \\ &= \int_{M_N f > 2^l} M_N f dx \to 0 \end{aligned}$$

Since $|g^l| \le c2^l$ we have $g^l \to 0$ as $l \to -\infty$ in the sense of distributions. Hence

$$f = \sum_l g^{l+1} - g^l = \sum_l b^l - b^{l+1}$$

in the sense of distributions. The difference $g^{l+1} - g^l$ is supported in

$$O^l = \{x : M_n f > 2^k\}$$

and

$$g^{l+1} - g^l = b^l - b^{l+1} = \sum_j (f - c^l_j)\eta^l_j - \sum_j (f - c^{l+1}_j \eta^{l+1}_j) = \sum_j A^l_j$$

with

$$A^l_j = (f - c^l_j)\eta^l_j - \sum_m (f - c^{l+1}_m)\eta^{l+1}_m \eta^l_j + \sum_m c_{j,m}\eta^{l+1}_m$$

with

$$c_{j,m} = \frac{\int (f - c^{l+1}_m \eta^l_j)\eta^{l+1}_m dx}{\int \eta^{l+1}_m dx},$$

since η^l_j is a partition of unity and hence

$$\sum_j c_{j,m} = 0.$$

Then

$$\int A^l_j dx = 0$$

$$\operatorname{supp} A^l_j \subset \tilde{Q}^{l,*}_j$$

$$|A^l_j| \le c2^l$$

by construction. We set

$$a_j^l = c^{-1} 2^{-l} |Q_j^l|^{-1} A_j^l$$

and

$$\lambda_j^l = c 2^l |Q_j^l|.$$

Then the a_j^l are atoms, and

$$\sum \lambda_j^l = c \sum 2^l |Q_j^l| = c \sum_l 2^l |\{M_N f > 2^l\}| \le c \int M_N f dx.$$

$\square$

It is not hard to see that $\mathcal{H}^1$ is a Banach space. We can use the atomic decomposition to define a norm:

$$\|f\|_{\mathcal{H}^1} = \inf\{\sum |\lambda_k| : \text{ there exists atoms with } f = \sum \lambda_k a_k\}. \tag{6.12}$$

It is a consequence that the span of atoms is dense in $\mathcal{H}^1$.

Corollary 6.10. *Let T be a Calderón-Zygmund operator. Then T defines a unique continuous operator from the Hardy space $\mathcal{H}^1$ to L^1. If T is a convolution operator satisfying the assumptions of the Mihlin-Hörmander theorem then T defines a unique continuous operator on $\mathcal{H}^1$.*

Proof. We only prove the first part. The second part is an exercise. Let a be an atom. We want to prove that

$$\|Ta\|_{L^1} \le c(n).$$

By translation invariance we may assume that the corresponding ball has center 0, and be rescalling we may assume that the radius is 1. Then

$$\|Ta\|_{L^{p_0}} \le c\|a|_{L^{p_0}} \le c(n)$$

where p_0 is the exponent of the Calderón-Zygmund operator. We use this bound on $B_2(0)$. Outside we argue as for the proof of the boundedness of Calderón-Zygmund operators.

Now let $f \in \mathcal{H}^1$. By the atomic decomposition

$$f = \sum \lambda_j a_j$$

We define

$$Tf = \sum \lambda_j T a_j$$

where the right hand side converges in L^1. There is no other choice for the definition, but wellposedness has to be proven. Suppose

$$f = \sum \lambda_j a_j = \sum \mu_j b_j$$

with atoms $(a_j), (b_j)$ and summable sequences λ_j and μ_j. We have to show that for $\varepsilon > 0$ there exists N_0 so that for all $t > 0$ and $N > N_0$

$$|\{x : |\sum_{j=1}^N \lambda_j T a_j - \sum_{j=1}^N \mu_j T b_j| > t\}| < \varepsilon/t. \tag{6.13}$$

Then

$$\sum_{j=1}^{\infty} \lambda_j T a_j = \sum_{j=1}^{\infty} \mu_j T b_j$$

follows. Inequality (6.13) follows from two properties:

(1) The weak type inequality for Calderón Zygmund operators

$$|\{x : |Tg(x)| > t\}| \leq \frac{c}{t} \|g\|_{L^1}$$

(2) The convergence of the partial sums in L^1.

By the convergence there exists for $\tilde{\varepsilon} > 0$ and N_0 so that

$$\| \sum_{j=1}^{\infty} \lambda_j a_j dx - \sum_{j=1}^{N} \lambda_j a_j \|_{L^1} + \| \sum_{j=1}^{\infty} \mu_j b_j dx - \sum_{j=1}^{N} \mu_j b_j \|_{L^1} < \tilde{\varepsilon}$$

and then

$$|\{x : | \sum_{j=1}^{N} \lambda_j T a_j - \sum_{j=1}^{N} \mu_j T b_j | > t\}| < c\varepsilon'/t.$$

□

3. Duality and BMO

DEFINITION 6.11. *Let $f \in L^1_{loc}(\mathbb{R}^n)$. The sharp maximal function is defined by*

$$f^\sharp(x) = \sup_{B_r(y) \ni x} |B_r(y)|^{-n} \int_{B_r(y)} \left| f(w) - |B_r(y)|^{-1} \int_{B_r(y)} f(z) dz \right| dw \in [0, \infty]$$

Properties

(1) $(f+g)^\sharp(x) \leq f^\sharp(x) + g^\sharp(x)$
(2) $f^\sharp(x) \leq 2Mf(x)$. Hence $\|f^\sharp\|_{L^p} \leq c_n \frac{p}{p-1} \|f\|_{L^p}$

DEFINITION 6.12. *We define BMO as the space of all function for which the (semi) norm*

$$\|f\|_{BMO} = \|f^\sharp\|_{sup}$$

is finite.

Certainly $L^\infty \subset BMO$. Moreover $\ln(|x|) \in BMO$.

THEOREM 6.13. *Let $L : \mathcal{H}^1 \to \mathbb{R}$ be a continous linear map. Then there exists $f \in BMO$ such that for every atom*

$$L(a) = \int a f dx, \tag{6.14}$$

$$\|f\|_{BMO} = \|L\|_{(\mathcal{H}^1)^*}.$$

Vice versa: let $f \in BMO$. Then (6.14) defines a continuous linear functional on $\mathcal{H}^1$.

PROOF. We prove the second part first. If a is an atom with ball B then

$$\int f a dx = \int_B (f - f_B) a dx \leq \| f - f_B \|_{L^1(B9)} \| a \|_{L^\infty} \leq f^\sharp(x_0).$$

thus for $f \in L^\infty$ and $g \in \mathcal{H}^1$ or $f \in BMO$ and $g \in \mathcal{H}^1_a$

$$\int f g dx \leq \| f \|_{BMO} \| g \|_H^1$$

This implies the second statement.

Now let L be a linear function on $\mathcal{H}^1$ of norm at most 1. let B be a ball. Then

$$L^2(B) \ni f \to L(f - f_B)$$

defines a linear functional on L^2 which is represented by a function g^B so that

$$L(f - f_B) = \int g^B f dx$$

and in particular $\int_B g^b = 0$. We search for a function g so that

$$g - g_B = g^B$$

for all balls B. Let $B \subset B'$ be two balls and g^B resp. $g^{B'}$ the functions constructed above. For $f \in L^2(B)$ with $\int f dx = 0$ we have

$$L(f) = \int_B f g^B dx = \int_{B'} f g^{B'} dx$$

thus for such f

$$\int_B (g^B - g^{B'}) f dx = 0.$$

Thus $g^B - g^{B'}$ is constant on B. We define

$$g = g^{B_1(0)}$$

if $|x| < 1$. Choose

$$c_R = g^{B_R(0)} - g$$

for $x \in B_1(0)$ and define

$$g(x) = g_{B_R(0)}(x) - c_R$$

for $|x| < R$ and $R \geq 1$. This gives a consistent choice, and by the considerations of the first part $g \in BMO$. $\square$