Mathematical Institute, Bonn
Professor: Prof. Dr. Koch
Assistant: Gennady Uraltsev
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Real and Harmonic Analysis

Exercise Sheet 12 | 21 Jan, 2015 | Turn in by: 10:00 28 Jan, 2015

Problem 34 (6 points)

The space $W^{1,p}(\mathbb{R}^n)$ consists of functions $f \in L^p(\mathbb{R}^n)$ such that all the distributional partial derivatives $\partial_{x_i} f$ are also in $L^p(\mathbb{R}^n)$. This space is a Banach space with the norm: $\|f|_{W^{1,p}} = \|f\|_{L^p} + \sum_{i=1}^{n} \left\|\partial_{x_i} f\right\|_{L^p}$.
Is $W^{1,n}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$ and is the inclusion continuous?

Hint: Use the Poincaré inequality for balls: $\fint_B |f - f_B|^p \leq C_B \|\nabla f\|^p_{L^p(B)}$. How does C_B depend on the radius of the ball?

Problem 35 (10 points)

1. *5 points* Are $f \in C_c^\infty(\mathbb{R}^n)$ such that $\int f = 0$ in $H^1(\mathbb{R}^n)$? Are they dense?
2. *5 points* Are $f \in C_c^\infty(\mathbb{R}^n)$ a dense subspace of $BMO(\mathbb{R}^n)$? Let $VMO(\mathbb{R}^n)$ be functions in $BMO(\mathbb{R}^n)$ such that setting $\omega_{(r)} = \sup_{x\in\mathbb{R}^n} \fint_{B(x,r)} \left| f - f_{B(x,r)} \right|$ we have that $\lim_{r\to 0} \omega(r) = \lim_{r\to\infty} \omega(r) = 0$. Is $C_0(\mathbb{R}^n)$ included in $VMO(\mathbb{R}^n)$? Are these functions dense therein?

Problem 36 (14 points)

We have seen that if f is a combination of atoms in $\mathcal{H}^1$ the integral $\int f(x)g(x)\mathrm{d}x$ with $g \in BMO(\mathbb{R}^n)$ is well defined. For a fixed g the map $f \mapsto \int f(x)g(x)\mathrm{d}x$ defined on $\mathcal{H}^1$ atoms extends to a bounded linear functional on $\mathcal{H}^1$.

1. *7 points* Is it true that $fg \in L^1_{loc}(\mathbb{R}^n)$ if $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$? If not give counterexamples in $\mathbb{R}$ and in $\mathbb{R}^2$.

   Hint: Remember that $\log(|x|) \in BMO(\mathbb{R}^n)$. Are compactly supported 0 mean L^1 functions in $\mathcal{H}^1$?

2. *7 points* Let $\Delta u = f$ on $\mathbb{R}^2$. Supposing that $f \in \mathcal{H}^1(\mathbb{R}^2)$, is it true that u is continuous?

All answers should be fully justified. The solution to a given point may depend on the solutions of the previous points:

if you didn't manage to prove a required statement you can still use it in the subsequent points.
Working in small groups is encouraged (no more than 4 people/group). You may hand in only one answer sheet per group. Put all the names on the sheet. During the exercise sessions students must know how to solve all the exercises done by their group, otherwise the assignment is void.