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Real and Harmonic Analysis

Exercise Sheet 10 | 17 Dec, 2014 | Turn in by: 10:00 17 Dec, 2014

Problem 28 (10 points)

A distribution $\Omega \in D'(S^{n-1})$ on $S^{n-1} \subset \mathbb{R}^n$ is a linear functional on $C^\infty(S^{n-1})$ such that there exists $N \in \mathbb{N}$ and $C \geq 0$ s.t. $\left|\Omega[\varphi]\right| \leq C \left\|\varphi\right\|_{C^N(S^{n-1})}$ for all $\varphi \in C^\infty(S^{n-1})$.

1. *5 points* Show that if $\Omega[1] = 0$ the expressions $\psi \mapsto \int_r \Omega[\psi(r\cdot)]\frac{\mathrm{d}r}{r}$ defines a temperate distribution $K_\Omega \in S'(\mathbb{R}^n)$ that is homogeneous of degree $-n$. Does the expression $\psi \mapsto \int_{\mathbb{R}+} \Omega[\psi(r\cdot)]\frac{\mathrm{d}r}{r}$ define a distribution in $S'(\mathbb{R}^n)$ if the condition $\Omega[1] = 0$ is omitted?
2. *5 points* Show that all distributions of homogeneous degree $-n$ are of the form $\psi \mapsto \int_{\mathbb{R}+} \Omega[\psi(r\cdot)]\frac{\mathrm{d}r}{r} + c\psi(0)$ for some $c \in \mathbb{C}$ and some $\Omega \in D'(S^{n-1})$ with $\Omega[1] = 0$.

Problem 29 (20 points)

We will give a different characterization of smooth translation-invariant Calderón-Zygmund kernels.

Definition 1. *Suppose that a kernel $K \in S'(\mathbb{R}^n)$ coincides with a C^∞ function on $\mathbb{R}^n \setminus \{0\}$ and it satisfies the following conditions:*

Size condition $|\partial_x^\alpha K(x)| \leq C_\alpha |x|^{-n-|\alpha|}$ *on $\mathbb{R}^n \setminus \{0\}$ for all multi-indexes α.*

Cancellation condition *For any $\varphi \in C_c^\infty(B(0,1))$ s.t. $\|\varphi\|_{C^1} \leq 1$ we have*

$$\left|K[\varphi_R]\right| \leq C \quad \forall R > 0 \qquad \text{where } \varphi_R(x) = \varphi\left(\frac{x}{R}\right).$$

We say that that K is a smooth Calderón-Zygmund convolution kernel.

Definition 2. *Let $m \in L^\infty$ be function that on $\mathbb{R}^n \setminus 0$ is smooth and satisfies the condition*

Size condition $|\partial_\xi^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|} \quad \forall \xi \in \mathbb{R}^n \setminus 0.$

We refer to such a function as a smooth Mihlin multiplier.

We want to prove that K is a smooth Calderón-Zygmund kernel if and only if the multiplier of the associated convolution operator is a smooth Mihlin multiplier. For ease of notation suppose $n = 1$.

Hint: In most of these points suppose that K has compact support by applying a suitable cutoff function. Then reason by density/continuity

1. *5 points* Let K be a smooth Calderón-Zygmund convolution kernel and m its multiplier. Show that $m \in L^\infty(\mathbb{R}^n)$.

   Hint: Consider a cutoff function $\eta \in C_c^\infty\left(B(0,1)\right)$ s.t. $\eta(B(0,1/2)) = 1$ and represent $K = K_1 + K_2 = K(\cdot)\eta(\rho^{-1}\cdot) + K(\cdot)\left(1 - \eta(\rho^{-1}\cdot)\right)$ with $\rho > 0$ a parameter. Prove the bounds for K_1 and K_2 and optimize the parameter ρ to obtain the needed result.

2. *3 points* Show that m is a Mihlin multiplier. *Hint:* Mimic the procedure above

3. *4 points* Let m be a smooth Mihlin multiplier and let K be the convolution kernel of the operator associated to m. Prove that K coincides with a smooth function away from 0 and it satisfies the size conditions of a smooth Calderón-Zygmund kernel.

4. *4 points* Prove a weaker version of the cancellation condition for the kernel K above: show that for some sufficiently large $k \in \mathbb{N}$, we have $|K[\varphi_R]| \leq C$ for all $\varphi \in C_c^\infty\left(B(0,1)\right)$ such that $\|\varphi\|_{C^k} < 1$

5. *4 points* Suppose that a convolution kernel K coincides with a smooth function away from the origin and it satisfies the size conditions of a smooth Calderón-Zygmund kernel. Furthermore suppose that the weaker cancellation condition (above) holds some order of smoothness $k > 0$.

   Show that K satisfies the cancellation condition of a smooth Calderón-Zygmund convolution kernel.

   Hint: Do this by showing that if $\varphi \in C_c^\infty(B(0,1))$, $\|\varphi\|_{C^1} \leq 1$ then $\varphi(x) = \varphi(0)\eta(x) + x\psi(x)$ for some smooth cutoff function $\eta(x)$ independent of φ and some $\psi \in C_{b,c}(B(0,1))$ with $\|\psi\|_{sup} < C$. Notice that $xK(x)$ coincides with an L^1_{loc} function.

Notice that this exercises illustrates how a weak “mean zero” or cancellation condition on a convolution kernel is closely related to the L^2 boundedness of the associated operator.

All answers should be fully justified. The solution to a given point may depend on the solutions of the previous points: if you didn’t manage to prove a required statement you can still use it in the subsequent points.
Working in small groups is encouraged (no more than 4 people/group). You may hand in only one answer sheet per group. Put all the names on the sheet. During the exercise sessions students must know how to solve all the exercises done by their group, otherwise the assignment is void.