Mathematical Institute, Bonn
Professor: Prof. Dr. Koch
Assistant: Gennady Uraltsev
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Real and Harmonic Analysis

Exercise Sheet 9 | 10 Dec, 2014 | Turn in by: 10:00 17 Dec, 2014

Problem 25 (10 points)

Definition 1 (Hilbert-Schmidt operator)**.** *Given a separable Hilbert space $\mathcal{H}$ we say that a linear map $A : \mathcal{H} \to \mathcal{H}$ is a Hilbert-Schmidt operator if $\|A\|_{HS} < \infty$ where*

$$\|A\|_{HS}^2 = \sum_i \|Ae_i\|^2$$

where e_i is some orthonormal basis of $\mathcal{H}$. One can check that $\|\cdot\|_{HS}$ does not depend on the choice of the basis.

1. *3 points* Consider a kernel $K \in S'(\mathbb{R}^n \times \mathbb{R}^n)$ and the associated operator $T_K : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ given by $(T_K\varphi)[\psi] = K\left[\varphi(x)\psi(y)\right]$. Prove that K is a Hilbert-Schmidt operator if and only if $K \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$.

2. *1 points* Supposing $a \in S'(\mathbb{R}^n \times \mathbb{R}^n)$, let $a_n \to a$ in $S'(\mathbb{R}^n \times \mathbb{R}^n)$ Show that $Op_t a_n \varphi \to Op_t a \varphi$ in $S(\mathbb{R}^n)$.

3. *5 points* Show that $a^w \in HS\left(L^2(\mathbb{R}^n)\right)$ if and only if $a \in L^2\left(\mathbb{R}^n \times \mathbb{R}^n\right)$. What is the relationship between $\|a\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}$ and $\|a^w\|_{HS}$?

   Hint: Find the kernel K associated to a^w.

4. *1 points* Let $a \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and $a_R(x,\xi) = a(x,\xi)\eta\left({}^x/_R\right)\eta\left({}^\xi/_R\right)$ with $\eta \in C_c^\infty\left(B(0,1)\right)$ and $\eta = 1$ on $B(0, {}^1/_2)$. Show that $a_R^w \to a^w$ in $\|\cdot\|_{HS}$.

Problem 26 (10 points)

Definition 2. *Given $m \in \mathbb{R}$ and $\rho, \delta \in [0,1]$ we define the symbol classes $S^m_{\rho,\delta}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ as functions $a(x,\xi)$ satisfying*

$$\left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \leq C_{\alpha,\beta}\left(1 + |\xi|\right)^{m - \rho|\alpha| + \delta|\beta|}$$

for all α and β n-multi-indexes.

With this definition we want to express the basic rules of symbolic calculus. We consider only the cases $\delta = 0$ and $\rho = \{1, 0\}$.

1. *5 points* Let $a \in S^{m_1}_{\rho,\delta}$ and $b \in S^{m_1}_{\rho,\delta}$. Show that the bilinear map $(a, b) \mapsto a\#b$ is well defined and $a\#b \in S^{m_1+m_2}_{\rho,\delta}$. Notice (without proof) that the product formula holds for any $\rho \geq \delta$.

2. *3 points* Express the first three terms of the asymptotic expansion of $a\#b$.

3. *2 points* Show that symbols in the class $S^0_{1,0}$ define bounded operators on L^2.

Problem 27 (10 points)

Definition 3. *The Sobolev spaces $H^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$ are the spaces of tempered distributions f such that*

$$\|f\|_{H^s} = \left\| \widehat{f}(\xi) \left(1 + |\xi|^2\right)^{s/2} \right\|_{L^2} < \infty$$

1. *2 points* Show that functions in $H^s(\mathbb{R}^n)$ are locally integrable for $s \geq 0$. What is the dual of $H^s(\mathbb{R}^n)$? Show that functions in $H^k(\mathbb{R}^n)$ with $k \in \mathbb{N}$ are L^2 functions with all weak partial derivatives of order up to k also in L^2.

2. *4 points* Prove that if $a \in S^m_{1,0}$ with $m \in \mathbb{R}$ then a^w maps $H^s(\mathbb{R}^n)$ to $H^{s+m}(\mathbb{R}^n)$.

   Hint: Consider the operator $\left(1 + |D|^2\right)^{(s+m)/2} a^w(x, D) \left(1 + |D|^2\right)^{-s/2}$.

3. *4 points* We call $a \in S^m_{1,0}$ elliptic if $|a(x, \xi)| \geq C \left(1 + |\xi|^2\right)^{m/2}$ for some $C > 0$. Show that $a^{-1} \in S^{-m}_{1,0}$ if $a \in S^m_{1,0}$ and it is elliptic. What can you say about $\left(a^{-1}\#a\right) - 1$?

All answers should be fully justified. The solution to a given point may depend on the solutions of the previous points: if you didn't manage to prove a required statement you can still use it in the subsequent points.
Working in small groups is encouraged (no more than 4 people/group). You may hand in only one answer sheet per group. Put all the names on the sheet. During the exercise sessions students must know how to solve all the exercises done by their group, otherwise the assignment is void.