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Real and Harmonic Analysis

Exercise Sheet 8 26 Nov, 2014 Turn in by: 10:00 3 Dec, 2014

Problem 22 (10 points)

1. *2 points* Consider the measure $\mathrm{d}\mu$ on the segment $[-1,1]\times 0 \subset \mathbb{R}^2$ given by $\mu(A) = m^1\left(x \in [-1,1] \,|\, (x,0) \in A\right)$. Does $\widehat{\mu}$ decay at infinity?

2. *3 points* Consider the measure $\mathrm{d}\mu$ on the piece of the parabola $y = x^2$, $|x| < 1$ for $(x,y) \in \mathbb{R}^2$ given by $\int f \mathrm{d}\mu = \int_{[-1,1]} f(x,x^2)\mathrm{d}x$. Let $(\xi,\eta) \in \mathbb{R}^2$ be the Fourier coordinates of (x,y). Does $\widehat{\mu}$ decay at infinity? If so what is the rate of decay in the region $|\xi| \leq C|\eta|$ and $|\xi| > C|\eta|$ depending on $C > 0$?

3. *2 points* Consider the surface measure σ on $S^{n-1} \subset \mathbb{R}^n$ normalized so that $\sigma(S^{n-1}) = nm^n\left(B(0,1)\right)$. Show that $|\widehat{\sigma}(\xi)| \lesssim |\xi|^{-(n-1)/2}$ when $|\xi| \to \infty$.

4. *3 points* Find the asymptotics of $\widehat{\sigma}(\xi)$ up to second order i.e. find c_1, bounded $f_2(|\xi|)$, and $\alpha < 0$ s.t. $\sigma(\xi) = c_1 \cos(2\pi|\xi|)|\xi|^{-(n-1)/2}\left(1 + f_2(|\xi|)|\xi|^{\alpha} + o(|\xi|^{\alpha})\right)$ as $|\xi| \to \infty$.

Problem 23 (8 points)

We begin by proving Lemma 4.15 of the lecture so do not refer to it without proof.

Definition 1. *$S(t)$ is a unitary group if on a Hilbert space $\mathcal{H}$ if $S(t) \in U(\mathcal{H})$ for all t; $S(t+s) = S(t)S(s)$ and $t \mapsto S(t)u$ is continuous for all $u \in \mathcal{H}$.*

1. *2 points* Let u be a solution of $i\partial_t u + \Delta u = 0$ in $S'(\mathbb{R}\times\mathbb{R}^n)$. Prove that the space-time Fourier transform $\widehat{u}$ is supported on the set $\left\{(\tau,\xi) |\, \tau = -2\pi|\xi|^2\right\}$. Let $u_t = S(t)u_0$ with $u_0 \in L^2(\mathbb{R}^n)$. Show that $u \in S'(\mathbb{R}\times\mathbb{R}^n)$ and if u satisfies the Schroedinger equation in $S'(\mathbb{R}\times\mathbb{R}^n)$ then $u(\varphi) = \int_{\mathbb{R}^n} \left(\mathcal{F}_x u_0\right)(\xi)\, \widehat{\varphi}\left(-2\pi|\xi|^2, \xi\right) \mathrm{d}m^n(\xi)$ where $\mathcal{F}_x$ is the Fourier transform along $x \in \mathbb{R}^n$ and $\widehat{\varphi}$ is the space-time Fourier transform of φ on $\mathbb{R}\times\mathbb{R}^n$.

2. *3 points* Consider the restriction map R of the Fourier transform on $\mathbb{R}\times\mathbb{R}^n$ to the parabola $\tau = -2\pi|\xi|^2$ given for $f \in S(\mathbb{R}\times\mathbb{R}^n)$ by $f \mapsto \widehat{f}(-2\pi|\xi|^2, \xi)$. What is the relationship between R and the initial value problem solution map $u_0(x) \mapsto \left(S(t)u_0\right)(x)$?

3. *3 points* Using Strichartz estimates prove the Thomas-Stein Restriction Theorem for the parabola i.e. that R is a bounded operator from $L^p(\mathbb{R}^{n+1}) \to L^2\left((-2\pi|\xi|^2, \xi) \subset \mathbb{R} \times \mathbb{R}^n, \mathrm{d}\sigma\right)$ with $p = \frac{2(n+2)}{n+2}$ and with $\mathrm{d}\sigma$ the pushforward measure on the parabola via the map $\xi \mapsto (-2\pi|\xi|^2, \xi)$.

Problem 24 (12 points)

We want to study the solutions to the linearized KDV equation on $\mathbb{R} \times \mathbb{R}$:

$$\begin{aligned} \partial_t u + \partial_{xxx} u &= 0 \\ u(0, \cdot) &= u_0(\cdot). \end{aligned}$$

1. *2 points* Show that solutions u formally satisfy $u(t,x) = g_t * u_0$ where the convolution occurs on the spatial variable. Show that $g_t(x) = \left(4\pi^2 t\right)^{-1/3} Ai\left(\frac{x}{\left(4\pi^2 t\right)^{1/3}}\right)$ for some distribution $Ai \in S'(\mathbb{R})$. Prove that given $u_0 \in L^2(\mathbb{R})$ the solution is given by $u(t,x) = \left(S(t)u_0\right)(x)$ for some unitary group $S(t)$. (Notice that Ai here may be normalized differently than in the lecture).

2. *5 points* g_t are kernels defined a priori as distributions. We want to show that they actually coincide with smooth bounded functions. Show that $\widehat{Ai}(\xi) = e^{2\pi i \xi^3}$. Prove that $Ai \in C^\infty(\mathbb{R})$ and that $\forall k \in \mathbb{N} \cup \{0\}$ we have $\|Ai\|_{C^k(\mathbb{R})} \leq C_k$.

   Hint: Show that $Ai(\varphi) = \lim_{\varepsilon \to 0} \left[\mathcal{F}^{-1}\left(e^{2\pi i \xi^3} e^{-\varepsilon \xi^2}\right)(\varphi)\right]$

3. *5 points* Prove that $|Ai(x)| = O(|x|^{-\frac{1}{4}})$ for $|x| \to \infty$.

   Hint: Separate the case $x > 0$ and $x < 0$. In the case $x < 0$ separate the integral into a piece close to the critical point c of the phase and an other piece far from it via multiplication by a cutoff fuction $\eta(\cdot - c)$

All answers should be fully justified. The solution to a given point may depend on the solutions of the previous points: if you didn't manage to prove a required statement you can still use it in the subsequent points.
Working in small groups is encouraged (no more than 4 people/group). You may hand in only one answer sheet per group. Put all the names on the sheet. During the exercise sessions students must know how to solve all the exercises done by their group, otherwise the assignment is void.