Mathematical Institute, Bonn
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Real and Harmonic Analysis

Exercise Sheet 6 | 19 Nov, 2014 | Turn in by: 10:00 26 Nov, 2014

Problem 19 (10 points)

Calculate the Fourier transform of the following functions/distributions:

- The Heaviside function $\mathbb{1}_{[0,+\infty)}$ on $\mathbb{R}$;
- $\frac{\sin(x)}{x}$ on $\mathbb{R}$;
- $e^{2\pi i k x}$ for some $k \in \mathbb{R}$;
- $P.V.\frac{1}{x}$ i.e. the distribution $\varphi \mapsto \lim_{\varepsilon \to 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \varphi(x)\frac{1}{x}\mathrm{d}x$;
- $\frac{1}{x-z}$ for some $z \in \mathbb{C}\setminus\mathbb{R}$;
- Show that the limit $\lim_{\varepsilon>0,\varepsilon\to 0} \frac{1}{x+i\varepsilon}$ defines a distribution on $\mathbb{R}$ and compute its Fourier transform.

Problem 20 (6 points)

Consider harmonic functions u on the half space $(0,\infty) \times \mathbb{R} \ni (t,x)$. After a Fourier transform with respect to x they satisfy at least formally

$$\hat{u}_{tt} - 4\pi^2|\xi|^2\hat{u} = 0.$$

Determine the bounded solution $\hat{u}$ with bounded boundary data $\hat{g}$ ($\hat{u}(0,\xi) = \hat{g}(\xi)$) and use the computation above to derive a formula for the Poisson kernel.

Problem 21 (4 points)

Show that the Hilbert transform on $\mathbb{T}$ and the operator $\varphi(x) \mapsto \lim_{\varepsilon\to 0} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \varphi(x-y)\frac{\mathrm{d}y}{y}$ do not extend to bounded operators on $L^\infty(\mathbb{T})$ and $L^\infty(\mathbb{R})$ respectively. Find bounded functions which are mapped to unbounded functions.

Problem 22 (8 points)

1. *3 points* Prove that a homogeneous distribution is tempered.

2. *5 points* Is the distribution δ_0 on $\mathbb{R}^N$ a homogeneous distribution? If so of what degree? Is the distribution $\partial_{x_k}\delta_0$ homogeneous? If so of what degree? Suppose T is a homogeneous distribution. Is $\partial_{x_j}T$ homogeneous?

All answers should be fully justified. The solution to a given point may depend on the solutions of the previous points: if you didn't manage to prove a required statement you can still use it in the subsequent points.
Working in small groups is encouraged (no more than 4 people/group). You may hand in only one answer sheet per group. Put all the names on the sheet. During the exercise sessions students must know how to solve all the exercises done by their group, otherwise the assignment is void.