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Real and Harmonic Analysis

Exercise Sheet 6 | 12 Nov, 2014 | Turn in by: 10:00 19 Nov, 2014

Problem 16 (8 points)

Given a sequence $(m_n \in \mathbb{C})_{n\in\mathbb{Z}}$, the so-called multiplier operator associated to m is given by

$$T_m f(x) = \sum_{n\in\mathbb{Z}} e^{2\pi i n x} m_n \widehat{f}(n)$$

and is defined for those functions $f \in L^1(\mathbb{T})$ such that $m_n \widehat{f}(n)$ is absolutely summable.

1. *5 points* Show that the Hilbert transform is a multiplier operator. What is its multiplier? Now suppose that $m_n \in BV(\mathbb{Z})$ i.e. $\sum_n |m_n - m_{n-1}| < \infty$. For which $p \in [1,\infty]$ does T_m extend to a bounded operator on $L^p(\mathbb{T})$?

   Hint: Express the multiplier operator associated to $\mathbb{1}_{[N,\infty)}$ in terms the Hilbert transform and the modulation symmetries $f(x) \mapsto e^{2\pi i k x} f(x)$

2. *3 points* Show that all bounded translation-invariant operators on $L^2(\mathbb{T})$ are multiplier operators. A projection operator P on a Hilbert space H is a self-adjoint operator such that $P^2 = P$. Characterize the multipliers of translation invariant projection operators.

Problem 17 (7 points)

Let us define the Radon transform of functions $\varphi \in S(\mathbb{R}^N)$.

$$\mathcal{R}\varphi(\pi) = \int_\pi \varphi(x)\, \mathrm{d}m^{N-1}(x)$$

where π is any $(N-1)$-dimenstional hyper-plane. One could immagine that one is scanning a N-dimensional object, like doing a magnetic resonance, so as a result one doesn't have pointwise measurements but only the total mass of matter in a given plane. Notice that the hyper-planes of $\mathbb{R}^N$ can be parametrized using a double covering map from $S^{N-1} \times \mathbb{R}$ via $(\omega, \rho) \in S^{N-1} \times \mathbb{R} \mapsto \{x \in \mathbb{R}^N |\ \langle x; \omega\rangle = \rho\}$ and this map naturally endows the space of hyper-planes with a structure of a differentiable manifold.

1. *5 points* We want to prove that a function φ can be recovered from its Radon transform. To do this express the Fourier transform $\widehat{f}(\xi)$ in terms of the Radon transform $\mathcal{R}f(\omega, \rho)$.

   Hint: Notice that in the integral definition of $f(\xi)$ the phase factor is constant on hyper-planes.

2. *2 points* Show that $\mathcal{R}$ is injective.

Problem 18 (15 points)

1. *5 points* Let us generalize Dirichlet sums to multi-dimensional tori $\mathbb{T}^d$. Given a function f on $\mathbb{T}^d$ its Fourier transform is $\widehat{f}(\boldsymbol{n}) = \iiint_{\mathbb{T}^d} f(\boldsymbol{x}) e^{-2\pi i \boldsymbol{n}\boldsymbol{x}} \mathrm{d}\boldsymbol{x}$ where $\boldsymbol{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ and $\boldsymbol{n}\boldsymbol{x} = \sum_{k=1}^d n_k x_k$. The partial sums are:
$$S_{\boldsymbol{N}} f(x) = \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^d \\ |n_k|\le N_k}} \widehat{f}(\boldsymbol{n}) e^{2\pi i \boldsymbol{n}\boldsymbol{x}},$$
where $\boldsymbol{N} = (N_1, \dots N_d) \in \mathbb{Z}^d$. Show that if $f \in L^p(\mathbb{T}^d)$ and $\boldsymbol{N}_l$ are a sequence of multi-indexes such that for each k^{th} component, $(N_l)_k \to +\infty$ as $l \to \infty$ then $S_{\boldsymbol{N}_l} f \to f$ in $L^p(\mathbb{T}^d)$.

2. *5 points* We have seen that Hardy functions $F \in H^1(\mathbb{D})$ are such that the non-tangential maximal function F^* is bounded in $L^1(\mathbb{T})$. This is due to cancellation phenomena. We illustrate how a similar phenomenon can happen on $\mathbb{R}^N$. Let $\varphi \in C^\alpha(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and let $M_\varphi f(x) = \sup_{t>0} f * \varphi_t(x)$ where $\varphi_t = D_t^1 \varphi = t^{-n} \varphi\left({\cdot}/{t}\right)$. Notice that we do NOT put an absolute value on f. Show that whilst $\left|M_\varphi f\right| \lesssim Mf$ there are non-zero functions $f \in L^1(\mathbb{R}^N)$ such that $M_\varphi f \in L^1$.

   Hint: Consider a function f with mean $⨍ f = 0$.

3. *5 points* We can adapt the theory of complex Hardy spaces to the upper complex plane $\mathbb{H} = \left\{z = x + iy \middle| \, y > 0\right\}$. Check that the Möbius transformation $z \mapsto i\,{}^{1+z}\!/_{1-z}$ sends $\mathbb{D}$ to $\mathbb{H}$. What is the conjugate kernel Q_y of the Poisson kernel for $\mathbb{H}$? Show that for $f \in L^p(\mathbb{R}, \frac{1}{1+x^2}\mathrm{d}x)$ we have that $Q_y * f \to \mathcal{H}f$ a.e. where $\mathcal{H}f = P.V. \int_{\mathbb{R}} \frac{1}{\pi x'} f(x - x')\mathrm{d}x'$

All answers should be fully justified. The solution to a given point may depend on the solutions of the previous points: if you didn't manage to prove a required statement you can still use it in the subsequent points.
Working in small groups is encouraged (no more than 4 people/group). You may hand in only one answer sheet per group. Put all the names on the sheet. During the exercise sessions students must know how to solve all the exercises done by their group, otherwise the assignment is void.