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Real and Harmonic Analysis

Exercise Sheet 5 | 05 Oct, 2014 | Turn in by: 10:00 12 Nov, 2014

Problem 13 (10 points)

Recall that $H^p(\mathbb{D})$ are the holomorphic functions u such that $\|u\|_{H^p(\mathbb{D})} \stackrel{\text{def}}{=} \sup_{0<r<1} \left\|u(re^{2\pi i\cdot})\right\|_{L^p(\mathbb{T})} < \infty$.

1. *4 points* Given $u \in H^p(\mathbb{D})$ prove that $|u(z)| \leq C_p \frac{\|u\|_{H^p(\mathbb{D})}}{|1-z|^{1/p}}$
2. *2 points* Given $p \in [1, \infty]$, for which $\alpha > 0$ we have that $\frac{1}{(1-z)^\alpha} \in H^p(\mathbb{D})$?
3. *4 points* For $p \in [1, \infty]$, show that $H^p(\mathbb{D})$ is a closed subset of $L^p(\mathbb{D})$.

Problem 14 (10 points)

We say that a set of functions $\{f_i\}_{i\in\mathcal{I}} \subset L^1(\mathbb{T})$ is uniformly integrable if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any set $A \subset \mathbb{T}$ with $|A| \leq \delta$ we have $\int_A |f_i| < \varepsilon$ for all $i \in \mathcal{I}$.

1. *5 points* Let $f_n \in L^1(\mathbb{R})$ be a sequence of uniformly integrable functions such that $\int_{\mathbb{R}} f_n(x)\varphi(x)\mathrm{d}x$ converges for all continuous bounded functions $\varphi \in C_b(\mathbb{R})$. Prove that there exists an $f \in L^1(\mathbb{R})$ such that $f_n \to f$ weakly in L^1.
2. *5 points* Let u be a real harmonic function on the unit disk $\mathbb{D} \subset \mathbb{C}$ such that $\sup_{0<r<1} \left\|u(re^{2\pi i\cdot})\right\|_{L^1(\mathbb{T})} < \infty$. Show that $u(re^{2\pi i\cdot}) = P_r * \mu(\cdot)$ with $\mu \in \mathcal{M}(\mathbb{T})$ a measure absolutely continuous with respect to the Lebesgue measure on $\mathbb{T} = \partial\mathbb{D}$ if and only if $\left\{u(re^{2\pi i\cdot})\right\}_{r\in(0,1)}$ are uniformly integrable.

Problem 15 (10 points)

Suppose that $f(x, y)$ is a subharmonic function on an open set $U \subset \mathbb{R}^2$ and let $u + iv$ be a holomorphic function on an open set $\Omega \subset \mathbb{C}$ with u and v real-valued and such that $(u, v)(\Omega) \subset U$. Show that $f(u, v)$ is subharmonic.

Hint: First show the statement for $f \in C^2$.

All answers should be fully justified. The solution to a given point may depend on the solutions of the previous points:

if you didn't manage to prove a required statement you can still use it in the subsequent points.
Working in small groups is encouraged (no more than 4 people/group). You may hand in only one answer sheet per group. Put all the names on the sheet. During the exercise sessions students must know how to solve all the exercises done by their group, otherwise the assignment is void.