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Real and Harmonic Analysis

Exercise Sheet 4 | 29 Oct, 2014 | Turn in by: 8:00 5 Nov, 2014

Problem 10 (13 points)

We want to study Lorentz spaces more carefully. Recall that for $p \in (1;\infty)$ and $q \in [1;\infty]$ we define the Lorentz quasi-norms as $\|f\|_{L^{p,q}} = p^{\frac{1}{q}} \left\| \lambda\, \mu\left(\{x|\ |f(x)| > \lambda\}\right)^{1/p} \right\|_{L^q\left(\mathbb{R}^+,\, \frac{d\lambda}{\lambda}\right)}$.

We are working on a sigma-finite measure space $(\mathbb{X}, \mu)$.

1. *4 points* We can discretize the integral that appears in the definition of the quasi-norm. Show that

$$p^{\frac{1}{q}} \left(\sum_{k\in\mathbb{Z}} 2^{kq} \mu\left(\left\{x|\ |f(x)| > 2^k\right\}\right)^{q/p} \right)^{\frac{1}{q}}$$

defines an equivalent quasi-norm to $\|\cdot\|_{L^{p,q}}$. Is the equivalence constant uniform in p and q? Let $f_k = f\,\mathbb{1}_{\left\{x|\ |f(x)|\in\left[2^k;\,2^{k+1}\right)\right\}}$ for $k \in \mathbb{Z}$, show that

$$\|f\|_{L^{p,q}} \approx_{p,q} \left\| \left\| f_k(x) \right\|_{L^p(x\in\mathbb{X},\mu)} \right\|_{l^q(k\in\mathbb{Z})}.$$

We write $A \approx_p B$ if there exists a constant $C_p > 0$ depending on the parameter p such that $C_p^{-1} B \le A \le C_p B$.

2. *5 points* Given $(p_0, q_0), (p_1, q_1) \in (1;\infty) \times [1;\infty]$ show that if $p_0 = p_1$ and $q_0 \le q_1$ then $L^{p_0,q_0} \subset L^{p_1,q_1}$ and the inclusion is continuous. Show that, in general, there are no other inclusions by providing a counterexample for any other two pairs of indexes. If $(\mathbb{X}, \mu)$ is a finite measure space (assume for simplicity $\mu(\mathbb{X}) = 1$) then what are the additional inclusions between spaces $L^{p,q}$?

3. *4 points* Suppose that $f \in L^{p_0}_w(\mathbb{X},\mu) \cap L^{p_1}_w(\mathbb{X},\mu)$ for some (not necessarily finite) measure space and $p_{0,1} \in (1,\ \infty]$. Show that $f \in L^{p_\theta,1}(\mathbb{X},\mu)$ for $p_\theta^{-1} = (1-\theta)p_0^{-1} + \theta p_1^{-1}$ and $\|f\|_{L^{p_\theta,1}} \lesssim_\theta \|f\|^{1-\theta}_{L^{p_0,\infty}} \|f\|^{\theta}_{L^{p_1,\infty}}$.

Problem 11 (7 points)

We have seen in the previous exercise sheet that if $Mf \in L^1(\mathbb{R}^N)$ then $f = 0$ because of non-integrability at infinity. Here we want to show a somewhat converse statement about local singularities of Mf. Suppose that we are on a torus $\mathbb{T} = {}^{\mathbb{R}}\!/\!_{\mathbb{Z}}$ of measure 1. Prove that

$$\|Mf\|_{L^1} \leq 1 + 2\int |f(x)| \log^+ |f(x)| \mathrm{d}x$$

where $\log^+ |x| = \log |x| \vee 0 = \max\{0; \log |x|\}$.

Hint: For $\|Mf\|_{L^1}$ use the bathtub representation (superlevel set integral representation): $\|g\|_{L^1} = \int_0^\infty \mu\left(\{x| \; |g(x)| > \lambda\}\right) \mathrm{d}\lambda$. Separate the case $\lambda > 1$ and $\lambda \leq 1$. Also separate $f = f^\lambda + f_\lambda$ where $f^\lambda = f \, \mathbb{1}_{\{x| \; |f(x)| \geq \lambda\}}$.

Problem 12 (10 points)

Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk $\{z| \; |z| < 1\}$. We want to prove that holomorphic functions on the unit disk are a closed subspace of $L^2(\mathbb{D})$.

1. *3 points* Show that a sequence of harmonic functions $f_n \in L^2(\mathbb{D})$ with bounded $L^2(\mathbb{D})$ norms are uniformly bounded in the $\sup$ norm on all compact subsets of $\mathbb{D}$.

   Hint: Use the mean value property.

2. *3 points* Prove the theorem.

3. *2 points* Let $f \in L^2(\mathbb{D})$ and let f° be the polar coordinate representation of f so that $f^\circ(\rho, \theta) = f(\rho e^{2\pi i \theta})$. Prove that Fourier transform along θ of f is a well-defined measurable function on $(0, 1) \times \mathbb{T}$:

$$\widehat{f_\rho}(n) \overset{\text{def}}{=} \int_{\mathbb{T}} f^\circ(\rho, \theta) e^{-2\pi i n \theta} \mathrm{d}\theta.$$

4. *2 points* Prove a variant of the Plancherel theorem:

$$\sum_{n \in \mathbb{Z}} \int_{(0,1)} \left|\widehat{f_\rho}(n)\right|^2 \rho \mathrm{d}\rho = \|f\|^2_{L^2(\mathbb{D})}$$

All answers should be fully justified. The solution to a given point may depend on the solutions of the previous points: if you didn't manage to prove a required statement you can still use it in the subsequent points.
Working in small groups is encouraged (no more than 4 people/group). You may hand in only one answer sheet per group. Put all the names on the sheet. During the exercise sessions students must know how to solve all the exercises done by their group, otherwise the assignment is void.