Mathematical Institute, Bonn
Professor: Prof. Dr. Koch
Assistant: Gennady Uraltsev
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Real and Harmonic Analysis

Exercise Sheet 3 | 22 Oct, 2014 | Turn in by: 8:00 29 Oct, 2014

Problem 7 (10 points)

1. *5 points* On $\mathbb{R}$ we define the dyadic intervals and the set of dyadic intervals

$$\mathcal{D} = \left\{ \left[a2^k;\ (a+1)2^k\right) \text{ with } a,k \in \mathbb{Z} \right\}.$$

   We define the dyadic maximal function:

$$M_{\mathcal{D}} f(x) = \sup_{\substack{I \in \mathcal{D} \\ x \in I}} \frac{1}{|I|} \int_I |f(y)| \mathrm{d}y.$$

   Show that $M_{\mathcal{D}} f$ is measurable for $f \in L^1_{loc}(\mathbb{R})$.
   For which $p \in [1,\infty]$ is $M_{\mathcal{D}}$ bounded on $L^p(\mathbb{R})$? Does $M_{\mathcal{D}}$ satisfy the weak bounds $\left| \left\{ x \middle| \ M_{\mathcal{D}} f(x) \geq \lambda \right\} \right| \leq C \frac{\|f\|_{L^1(\mathbb{R})}}{\lambda}$?

2. *5 points* Prove that if $f \in L^1_{loc}(\mathbb{R}^n)$ and if $Mf \in L^1(\mathbb{R}^n)$ where M is the (non-centered) maximal function, then $f = 0$. Prove the same statement for $M_{\mathcal{D}}$.

Problem 8 (13 points)

Consider the heat equation on $\mathbb{R}^N \times [0, +\infty)$

$$\begin{aligned} \partial_t u(x,t) - \Delta u(x,t) &= 0 \\ u(x,0) &= u_0(x) \end{aligned}$$

1. *5 points* Let $K_t = \frac{1}{\sqrt{4\pi t}^N} e^{-\frac{x^2}{4t}}$. Given $u_0 \in L^p(\mathbb{R}^N)$ define $u(x,t) = K_t * u_0(x)$ and prove that $u \in C^\infty\left(\mathbb{R}^N \times (0;\infty)\right)$ and that u satisfies the heat equation. For which p does u satisfy $u(\cdot,t) \to u_0$ in $L^p(\mathbb{R}^N)$ as $t \to 0$ almost everywhere?

2. *4 points* Let $G_t(x) = \sum_{k\in\mathbb{Z}} K_t(x+k)$. Show that $G_t(x)$ is a well defined function on $\mathbb{T}$ and that it is an approximate identity for $t \to 0$. Show that for any $p \in [1,\infty)$ and $u_0 \in L^p(\mathbb{T})$ the function $u(x,t) = G_t * u_0(x)$ is smooth and solves the heat equation on $\mathbb{T} \times (0;\infty)$. Furthermore show that $u(\cdot,t) \to u_0(\cdot)$ in $L^p(\mathbb{T})$ as $t \to 0$.

3. *4 points* Let $u \in C\left(\mathbb{T} \times [0;\, \infty)\right) \cap C^2\left(\mathbb{T} \times (0;\, \infty)\right)$ be a solution to the heat equation on $\mathbb{T}$.

   We define

   $$\widehat{u}_t(n) \overset{\text{def}}{=} \int_{\mathbb{T}} u(x,t) e^{-i2\pi n x} \mathrm{d}x.$$

   Express $\widehat{u}_t(n)$ in terms of u_0.

Problem 9 (7 points)

1. *3 points* For every $p \in [1, \infty)$ exhibit a function $f \in L^p_w(\mathbb{R}^N) \setminus L^p(\mathbb{R}^N)$.
2. *4 points* Prove that on a finite measure space $(\mathbb{X}, \mu)$ we have $L^q_w(\mathbb{X}) \subset L^p(\mathbb{X})$ with $1 \leq p < q \leq \infty$.

All answers should be fully justified. The solution to a given point may depend on the solutions of the previous points: if you didn't manage to prove a required statement you can still use it in the subsequent points.
Working in small groups is encouraged (no more than 4 people/group). You may hand in only one answer sheet per group. Put all the names on the sheet. During the exercise sessions students must know how to solve all the exercises done by their group, otherwise the assignment is void.